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School of Mathematical Sciences

Mathematics News from the University of Nottingham

Volume 1, Number 2 (March 2022)

In this issue of the University of Nottingham Mathematics Newsletter we have:

- **Studying mathematics at the University of Nottingham:** This issue we focus on our 3-year single-honours BSc degree in mathematics
- **Puzzle 1.2:** A mathematical puzzle about discs
- **Solution to Puzzle 1.1:** The solution to last issue's puzzle
- **Interesting Mathematical Facts**: Alice Roth and Swiss cheeses
- **Useful Links**: Links to useful resources
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We welcome feedback, comments and suggestions. Please let us know what you found most interesting, what else you would like to see, and any other comments you have by filling in the short feedback form at https://tinyurl.com/uonmathsnewsfeedback

Alternatively, you can contact me by email at joel.feinstein@nottingham.ac.uk

Joel Feinstein (Newsletter Editor)

Studying mathematics at the University of Nottingham

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The three-year Mathematics BSc at the University of Nottingham

Do you want to learn modern statistical methods and how to use them in applications? Are you interested in studying mathematical techniques and using them in medical, financial or engineering modelling? Do you want to know why satellite navigation systems depend in a crucial way on Einstein's theory of relativity? Are you excited about number theory and its applications to cryptography and internet security? These are just some of the areas of research our academics are working on.

Our three-year single-honours mathematics BSc (https://tinyurl.com/uonG100) gives you the opportunity to learn about pure and applied mathematics, probability and statistics from our internationally recognized experts. You can then put your knowledge into practice during a third-year group project under expert supervision. You might also want to spend some time studying abroad, or working with our academics on a genuine research project which could even lead to a publication in a refereed journal.

In your first year you will be supported through our award-winning Peer-Assisted Study Support (PASS) scheme. This is run by senior students, who provide support on a range of important first-year topics, and help students with the transition to university-level mathematics.

As you progress through the course, you can begin to specialise in the areas that you find most interesting and exciting. Along with the mathematics, you will also learn important skills which are highly valued by employers, such as analytical thinking and problem-solving. This opens the way to a wide variety of possible careers. Our graduates can use the skills and the mathematical principles and techniques they have developed for jobs in, for example, data analytics, engineering, finance, medicine and teaching. This course also provides a good foundation for further study such as an MSc.

Puzzle 1.2

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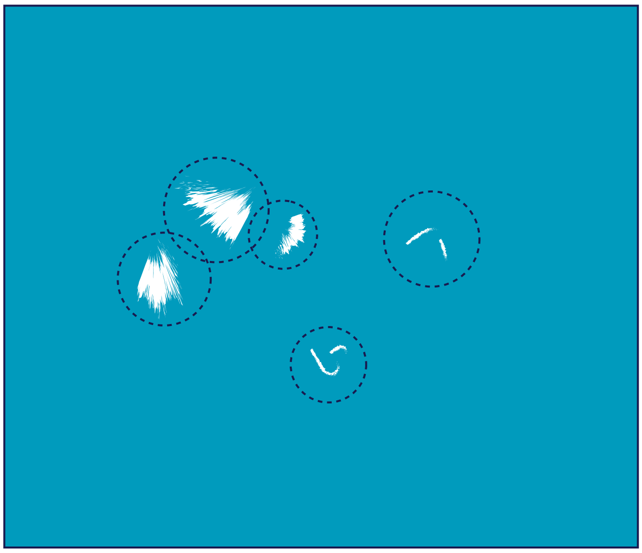
A Curious Carpet Conundrum

For this puzzle, you need to imagine that you have come across an unusual carpet shop, which we'll call **Berner Carpets**. They only sell circular carpets, which we'll call **discs.** They also have an unusual pricing system: they are happy to sell you a disc of any specified radius, measured in centimetres, and the price they charge is then £1 per centimetre. So if you want a disc whose **diameter** is 1 metre, the radius would be 50cm and so the price would be £50. They calculate the total cost of the discs you choose exactly, but then they round the total cost up or down to the nearest penny. If the total cost is less than £10, then they round the cost up. Otherwise they round the cost down. For example, if you want to buy some discs whose radii add to 45.238 cm, the price would be £45.23.

Paint marks (not to scale)

You come home after a day out to find that someone has left some paint marks on your floor. You have invited some important guests round, and you want to make a good impression. Unfortunately, you don't have time to remove the marks before your guests arrive. So as a temporary solution, you decide to use a carpet or carpets to cover the marks. However, the only carpet shop nearby that is open is Berner Carpets, and you can only afford to spend £40.

Now suppose that you have come up with a design which allows you to cover all of the marks using a few discs from Berner Carpets (finitely many, and more than one) and the sum of the radii of these discs is **exactly** 40cm.



Possible first design, with overlapping discs

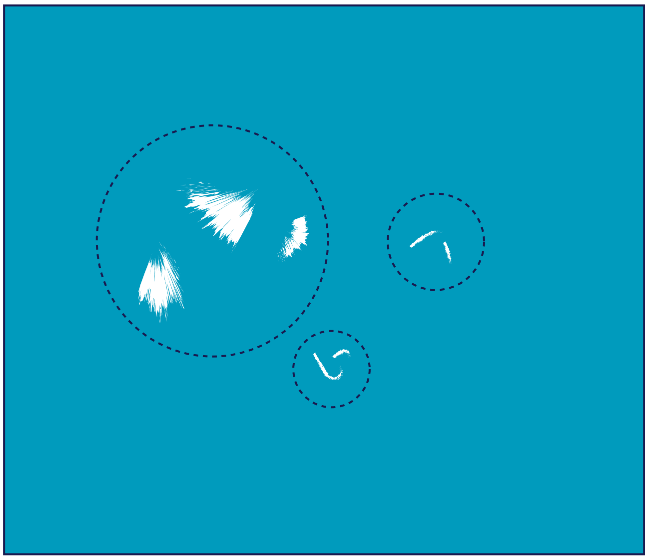
Unfortunately there is a snag: some of the discs in your design overlap each other, and that could lead to a tripping hazard.

Show that, in this situation, by modifying your design suitably, you can always find a way to cover the paint marks at a cost that is less than £40, while avoiding any overlaps between the discs. Show further that you can ensure that you don't have any pairs of discs which touch each other either.

It is acceptable to use just one disc if the price is under £40. But you are not allowed to cut the discs into pieces, as that would look bad.

You may assume that the discs in your original design are far enough away from the edges of your room that there is no danger of any disc that you need running up against the walls.

See below for an illustration of one possible modified design (not to scale), as well as the connection with Swiss cheeses, and some hints if you want them!



Possible modified design

The connection with Swiss cheeses

This puzzle is based on a geometrical result which my research students and I (Dr Feinstein) have used in our work on (mathematical) Swiss cheeses, and which also has other applications in **Complex Analysis**. See the article below on **Alice Roth and Swiss cheeses** to find out what these Swiss cheeses are, and which problem Alice Roth solved with them.

The puzzle becomes more challenging if you are allowed to work with **infinite** sequences of discs (in your initial design and in your modified design) so that you may have to add up infinitely many radii. This is a special case of one of the problems my research students and I solved. Although the basic idea remains the same, the process required to avoid any overlaps needs some care when you are working with infinite sequences of discs. If you also want to avoid having any touching discs in this setting, then you need to use more advanced mathematical techniques. For more on this and some other applications of Swiss cheeses (at research level), you can look at our survey article [FMY].

Hints for Puzzle 1.2

Remember that we are assuming that in your original design you planned to use finitely many discs (and more than one). The sum of the radii of those discs was exactly 40cm, and there was at least one pair of overlapping discs (not just touching). In your modified design, you want the sum of the radii of the discs to be strictly less than 40cm.

- First consider the special case where you have just two discs, one with radius 15cm and one with radius 25cm, and these two discs overlap. Show that you can cover these two discs and more using a single disc whose radius is strictly less than 40cm.
- Show more generally that you can cover any pair of overlapping discs using a disc whose radius is strictly less than the sum of the radii of the two smaller discs.
- Now suppose that you have two discs which just touch but do not overlap. Show that you can cover the two discs with a single disc whose radius is equal to the sum of the radii of the two smaller discs.
- How can you modify your original design systematically to ensure that you spend strictly less than £40, and you still cover all of the paint marks, but you avoid having any touching or overlapping discs?

References

[FMY] J. F. Feinstein, S. Morley and H. Yang, **Swiss cheeses and their applications**, Function Spaces in Analysis, 99-118, *Contemp. Math.*, 645, Amer. Math. Soc., Providence, RI, 2015, e-print available from https://tinyurl.com/SwissCheeseApplications

Solution to Puzzle 1.1

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Note: Here we will always assume that x and m are **positive integers** (i.e., they are whole numbers which are strictly greater than zero). We also assume that you know what a prime number is.

(a) Can $x^m + 1$ ever be prime?

Following the hint given last issue, we look at the special cases where $x = 1$ or $m = 1$.
When $x = 1$, we have $x^m + 1 = 2$ (whichever positive integer m we choose), and 2 is prime.
When $m = 1$, we have $x^m + 1 = x + 1$. We can get **every** prime number this way, and indeed we can get every positive integer except 1. For example, to obtain the prime number 13, we can take $x = 12$ and $m = 1$. These cases are not very interesting, and the answer to (a) is clearly “yes”.

(b) Can $x^m + 1$ ever be prime if $x > 1$ and $m > 1$?

Note that, if $x > 1$ and x is odd, then $x^m + 1 > 2$ and $x^m + 1$ is even, so we can’t get a prime number that way. Because of this, in (b) (and (c) below) we only need to check even numbers x.
There are certainly many examples when $m = 2$. For example, when $x = 2,\ 4,\ 6$, we get the prime numbers $2^2 + 1 = 5$, $4^2 + 1 = 17$ and $6^2 + 1 = 37$. The next one is $8^2 + 1 = 65$, and this is not prime. There are probably infinitely many x which work when $m = 2$, but no-one can prove this yet. This is one of Landau’s “unattackable” problems, as discussed last issue. But the answer to (b) is clearly still “yes”.

(c) Can $x^m + 1$ ever be prime if $x > 1$ and $m > 2$? (If so, what can you say about m?)

The suggestion last issue was to try a few values of x and m to see what examples of prime numbers you could find, with the warning that $x^3 + 1$ can’t be prime when $x > 1$.
Did you find some examples? Because of what we said above, you should only consider even numbers x. For example, $2^4 + 1 = 17$ and $2^8 + 1 = 257$, and 17 and 257 are both prime. These are examples of Fermat numbers and **Fermat Primes.** You may also have found the prime number $1297 = 6^4 + 1$. But you **won’t** have found any examples with $x > 1$ when $m = 3, 5, 6$ or 7. So the first few positive integers m which work for at least some $x > 1$ are $m = 1, 2, 4, 8$ and we might begin to suspect that m needs to be a power of 2 for us to have any chance. How can we be **sure** that this is correct?

Let’s start by looking at the above claim that $x^3 + 1$ can’t be prime when $x > 1$. We will show this using **factorization**. For **all** positive integers x, we have

$$x^3 + 1 = (x + 1)(x^2 - x + 1),$$

which you can check just by multiplying out the brackets. (We’ll have a look later at how you might spot this factorization.) So $x^3 + 1$ is always divisible by $x + 1$. This suggests that $x^3 + 1$ can **never** be prime. However, a little care is needed. We already know that when $x = 1$, we have $x^3 + 1 = 2$, and 2 **is** prime. In this case the fact that $x^3 + 1$ is divisible by $x + 1$ just tells us that 2 is divisible by 2, which is unhelpful. However, if $x > 1$, then we can safely claim that

$$1 < x + 1 < x^3 + 1,$$

and so the fact that $x^3 + 1$ is divisible by $x + 1$ really does show that $x^3 + 1$ is not prime.

How might we spot that we can factorize the polynomial $x^3 + 1$ this way? Well, it isn't too hard to spot that $(-1)^3 + 1 = 0$, i.e., -1 is a root of the polynomial $x^3 + 1$, and so $x + 1 = x - (-1)$ **must** be a factor of the polynomial $x^3 + 1$ (by, for example, the polynomial remainder theorem or the factor theorem). The good thing about this approach is that it tells us that $x + 1$ will be a factor of the polynomial $x^m + 1$ whenever m is **odd**, because then $(-1)^m + 1 = 0$, and so -1 is a root of the polynomial $x^m + 1$. In fact, when m is **odd** and $m > 3$, we have

$$x^m + 1 = (x + 1)(x^{m-1} - x^{m-2} + x^{m-3} - \cdots - x + 1).$$

- Note that this approach **doesn't** work when m is even, because then $(-1)^m + 1 = 2 \neq 0$.
- This equality is also related to facts about summing the terms of a geometric progression (with common ratio $r = -x$.

At this point, we know that $x^m + 1$ is divisible by $x + 1$ whenever m is odd. Now if $x > 1$, $m > 1$ and m is odd, then we have

$$1 < x + 1 < x^m + 1,$$

and so the fact that $x + 1$ is a divisor of $x^m + 1$ ensures that $x^m + 1$ is not prime.

At this point, with $x > 1$, we have eliminated all odd m with $m > 1$. But what about, for example, $m = 6$? Here the trick is to note that

$$x^6 + 1 = (x^2)^3 + 1.$$

Since $x^2 > 1$, our previous work tells us that $(x^2)^3 + 1$ can't be prime, and so $x^6 + 1$ is not prime.

Finally, if $m > 1$ and m is even, but m is not a power of 2, then we can write $m = 2^k q$ for some odd number $q > 1$ and some positive integer k. (To find q, keep dividing m by higher and higher powers of 2 until you eventually arrive at an odd number. For example, if $m = 96$, we find that we need $k = 5$ and $q = 3$, because $96 = 2^5 \times 3$.) This gives us

$$x^m + 1 = \left(x^{2^k}\right)^q + 1.$$

(Note here that x^{2^k} means x raised to the power 2^k, rather than x^2 raised to the power k.)

Since $x^{2^k} > 1$ and q is odd and greater than 1, our earlier results tell us that $\left(x^{2^k}\right)^q + 1$ can't be prime, and so $x^m + 1$ is not prime.

We have finally established the following, which is probably the best we can hope for.

> If $x > 1$ and $m > 1$, and $x^m + 1$ is prime, then m must be a power of 2.

However, this does not actually tell us whether **all** such m **do** give us some examples of primes this way. This turns out to be yet another "unattackable" problem, resembling Landau's fourth problem (as discussed last issue). Let's make this question precise.

Question: Is it true that, whenever m is a power of 2, then there is always at least one integer $x > 1$ such that $x^m + 1$ is prime?

In fact, it is probably true that, whenever m is a power of 2, then there are always **infinitely many** integers $x > 1$ such that $x^m + 1$ is prime. But proving that there is always **at least one** such x would be a good start!

Alice Roth and Swiss Cheeses

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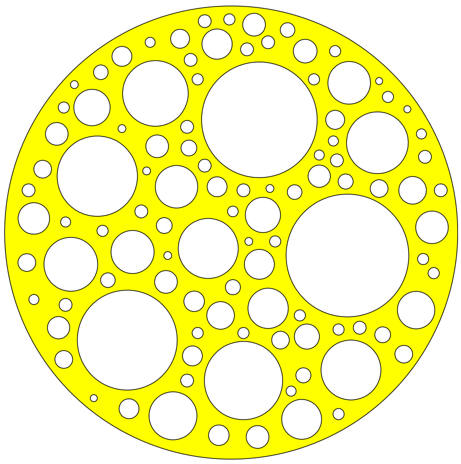
Alice Roth (1905-1977) was a brilliant Swiss mathematician who made important contributions to approximation theory and the theory of functions of a complex variable. The Swiss Federal Institute of Technology in Zürich (ETH Zürich) has recently introduced a new lecture series in her honour, the Alice Roth Lectures, which will run for the first time in March 2022.

Alice Roth in the 1920's
Copyright Roth Bollinger Family

As a university student, Roth studied at ETH Zürich (with main subject mathematics), obtaining a master's degree in 1930. She then spent ten years teaching in schools but, while still teaching, she completed a PhD thesis in 1938 under the supervision of George Pólya (who had also supervised her master's thesis). Her PhD thesis was so impressive that the ETH awarded her their silver medal. She was the first woman ever to receive this prestigious award.

It was in her PhD thesis where, among other things, Roth introduced (mathematical) Swiss cheeses in order to solve a major open problem in rational approximation [R]. The Weierstrass approximation theorem from the late 19th century had shown that continuous, real-valued functions on a closed interval in the real line could be approximated as closely as you like (uniform approximation) using polynomial functions with real coefficients. Since then, mathematicans had obtained a number of results on approximation of continuous, **complex-valued** functions on various plane sets (subsets of the complex plane). This included results on approximation by polynomial functions with complex coefficients, and by the corresponding rational functions (quotients of polynomial functions). However, for a certain class of plane sets (compact plane sets with empty interior), it was open whether or not it was always true that every continuous function on the set could be (uniformly) approximated using rational functions. Roth's Swiss cheese was the first example in this class of sets where some continuous functions could not be approximated in this way.

Roth's Swiss cheese was obtained by starting with a disc in the plane, and then punching out an infinite sequence of smaller discs, leaving circular holes in the original disc. The smaller discs neither touched nor overlapped with each other, and none of the smaller discs touched or overlapped with the outside of the original disc. By choosing the centres and radii of the holes in an ingenious way, Roth ensured that the sum of the radii of the smaller discs was strictly less than the radius of the starting disc, but that the resulting Swiss cheese had 'empty interior': it was impossible to choose any further discs to punch out from the Swiss cheese without overlapping at least one of the existing holes. In spite of this, the area of the Swiss cheese was still strictly positive. Roth proved that her Swiss cheese solved the rational approximation problem we described above.



Part way towards making a Swiss cheese, after punching out finitely many discs (not to scale)

Alice Roth in 1961
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From 1940 to 1971, Roth taught mathematics and physics at a private school in Bern. She was a very popular and influential teacher, but she was always interested in research. When she retired from teaching in 1971, she was able to return to research with great success. She published several papers, and she was invited to give talks on her work both in Switzerland and abroad. Unfortunately in 1976 she became ill, and she died in July 1977.

Acknowledgments

- I am very grateful to the Roth Bollinger family for their permission to use some of their photographs of Alice Roth.
- The historical information in this article is largely based on material from [DGGS], where you can find a far more detailed account of Alice Roth's life and work.
- I will always be grateful to Alice Roth for inventing Swiss cheeses, which are among my favourite examples in all of mathematics. I still use Swiss cheeses and their variants regularly in my own research, and my research students and I have even developed the theory of a space entirely populated by abstract Swiss cheeses [FMY2]!

Alice Roth in 1976
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References

[DGGS] Ulrich Daepp, Paul Gauthier, Pamela Gorkin, and Gerald Schmieder, **Alice in Switzerland: the life and mathematics of Alice Roth**, *Math. Intelligencer* 27 (2005), no. 1, 41–54.

[FMY2] J. F. Feinstein, S. Morley and H. Yang, **Abstract Swiss Cheese Space and Classicalisation of Swiss Cheeses**, *Journal of Mathematical Analysis and Applications* 438 (2016), 119-141: Open Access funded by EPSRC, http://dx.doi.org/10.1016/j.jmaa.2016.02.004

[R] A. Roth, **Approximationseigenschaften und Strahlengrenzwertemeromorpher und ganzer Funktionen**, *Comment. Math. Helv.* 11 (1938), 77–125.

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