SIMONS SYMPOSIUM 2015: OPEN PROBLEM SESSIONS

The 2015 Simons Symposium on Tropical and Nonarchimedean Geometry included two open problem sessions, in which leading experts shared questions and conjectures that they find interesting and important. One purpose of these sessions is to help set a direction for the future development of tropical and nonarchimedean geometry, and to give young researchers clear targets to aim for in their own research programs. What follows is an attempt to record as much of the essential content as possible from these open problem sessions.

1. Thuillier's deformation retraction in the imperfect case. (Johan de Jong)

In Thuillier's paper on toroidal and nonarchimedean analytic geometry, he associates to a toroidal structure on a scheme X over a trivially valued perfect field a skeleton $\Sigma \subset X^{\mathrm{an}}$, together with a strong deformation retraction $X^{\mathrm{an}} \to \Sigma$.

Problem: Can one define such a deformation retraction in the case where K is not perfect?

2. Homotopy types of dual complexes of log resolutions. (Antoine Ducros, Dan Abramovich)

As a consequence of the theorem mentioned above, Thuillier proves that the homotopy type of the dual complex of a log resolution of X is independent of the choice of a resolution. It is natural to ask about the following generalization:

Problem: Let X be a quasi-excellent scheme, and let $Y \to X$ be a log resolution of singularities. Is the homotopy type of the dual complex of $Y \to X$ independent of the choice of the log resolution?

Ideally, one would like an analogous intrinsic characterization of the homotopy type of the dual complex, in terms of Berkovich analytifications or $\beth$-spaces. One could also ask whether the *simple* homotopy type is independent of the choice of resolution. This is true in the classical complex geometry setting, where it follows from the Weak Factorization Theorem.

3. Existence of deformation retractions to the skeleton in more general situations. (Kiran Kedlaya)

Kiran Kedlaya has proved that if R is a perfect ring of characteristic p (e.g. the perfection of a finite type ring over a field), then there is a deformation retraction $\mathcal{M}(W(R)) \to \mathcal{M}(R)$, where $\mathcal{M}$ denotes the Berkovich spectrum and $W(R)$ denotes the ring of Witt vectors of R. The motivation for this kind of result comes from p-adic Hodge theory and the theory of perfectoid spaces.

Date: February 2 and 5, 2013.

Problem: Is there a common generalization of this theorem and the deformation retraction results established by Berkovich and Thuillier?

4. Brill–Noether existence for discrete graphs. (Sam Payne)

Matt Baker has conjectured that if the Brill–Noether number

$$\rho(g, r, d) = g - (r + 1)(g - d + r)$$

is non-negative then every finite graph G of genus g has a divisor of degree d and rank at least r (in the sense of Baker–Norine) supported on the vertices. The metric graph analogue of this result is known, but the discrete version remains open. (There seems to be a gap in the published proof by Caporaso.)

Problem: Prove or disprove the Brill–Noether existence conjecture for graphs.

Partial results in the affirmative direction have been reported by Cools and Draisma in rank 1.

5. Rational connectivity and contractibility. (Johannes Nicaise)

Morgan Brown and Tyler Foster have shown that the Berkovich analytification of a rationally connected variety over $\mathbf{C}((t))$ is contractible, and that if $f : X \to Y$ has rationally connected fibers then f^{an} is a homotopy equivalence. The proofs rely in an essential way on the Minimal Model Program (MMP).

Problem: Do similar results hold for rationally connected varieties over general non-Archimedean fields? Can one prove this without using MMP?

6. Essential skeleta without MMP. (Johannes Nicaise)

Nicaise and Xu have shown that if X is a smooth projective variety with semi ample canonical bundle over $\mathbf{C}((t))$ which comes by base change from the function field of a curve, then there is a deformation retraction from X^{an} to the *essential skeleton* of X^{an} in the sense of Kontsevich–Soibelman and Mustaţă–Nicaise. The essential skeleton is indeed therefore a "skeleton" in the more traditional sense of the word. The proof makes heavy use of MMP.

Problem: Does a similar result hold over more general non-Archimedean fields? Can one prove the above result without using MMP?

[Editor's note: Dan Abramovich pointed out that it would be good to have an understanding of the steps in MMP in terms of non-Archimedean geometry, and Matt Baker noted along these lines that it would be interesting to develop a "combinatorial" analogue of the MMP for certain classes of piecewise-linear spaces.]

7. Intrinsic description of skeleta. (Antoine Ducros, Michael Temkin)

For a curve X over a non-Archimedean field K, the essential skeleton coincides with the set of all points of X^{an} having no neighborhood isomorphic to a disk.

Problem: Let X be a smooth projective variety with semi-ample log-canonical bundle, and consider the set Σ' consisting of all points in X^{an} which have no neighborhood isomorphic to a relative disk over a smaller-dimensional space. Is this a skeleton for X^{an}, i.e., does it have a (canonical) piecewise-linear structure and is there a strong deformation retraction from X^{an} to Σ'? How does Σ' relate to the essential skeleton of X?

8. The Dirichlet problem on higher dimensional skeleta. (Joe Rabinoff)

In dimension 1, the restriction of $\log|f|$ to a skeleton is determined by $\mathrm{Trop}_*(\mathrm{div}(f))$. This essentially amounts to solving a discrete Dirichlet problem using Kirchhoff's laws. However, an example due to Dustin Cartwright shows that the corresponding statement does not hold for surfaces, even when the skeleton has pure dimension 2.

Problem: Find a reasonably general set of conditions guaranteeing that the restriction of $\log|f|$ to a skeleton is determined by $\mathrm{Trop}_*(\mathrm{div}(f))$.

Dustin Cartwright believes that this should hold when each locally closed stratum of the special fiber is affine or, more generally, when no stratum contains a positive dimensional proper subvariety.

9. Tropicalization of principal divisors. (Joe Rabinoff)

If X is a curve, every tropical principal divisor on a skeleton of X is the push-forward of a principal divisor on X (Baker-Rabinoff). The corresponding statement is not known in higher dimensions.

Problem: Let $\Sigma(\mathfrak{X})$ be the skeleton of a semistable formal model of a smooth proper variety X over a complete and algebraically closed non-trivially valued non-Archimedean field K. What is the image under $\mathrm{Trop}_* : \mathrm{Div}(X) \to \mathrm{Div}(\Sigma(\mathfrak{X}))$ of the set of principal divisors on X?

10. Comparing the complex and non-Archimedean theories for Monge-Ampère operators and pluripotential theory. (Matt Baker)

The framework developed by Berkovich produces analytic spaces over certain Banach rings that can interpolate between nonarchimedean analytic spaces and classical complex analytic spaces.

Problem: Formulate precisely and prove that the non-Archimedean Monge–Ampère operator is a degeneration of the complex Monge–Ampère operator, and that non-Archimedean PSH functions are degenerations of Archimedean ones.

It would also be interesting to look for other direct relationships between complex and nonarchimedean analytic phenomena, in the spirit of work by Berkovich on degenerations of Hodge structures and by Jonsson on limits of complex amoebas.

Johannes Nicaise mentioned that a possibly related construction is to start with X over $\mathbb{C}((t))$, take a simple normal crossing (or log smooth) model $\mathfrak{X}$ over $\mathbb{C}[[t]]$, and look at the special fiber $\overline{\mathfrak{X}}$ with its induced log structure. Then Kato and Nakayama construct the log space $|\overline{\mathfrak{X}}^{\log}|$ which recovers the topology of the general fiber, when X comes from a family of varieties over a punctured disc.

11. Degenerations of Calabi-Yau manifolds. (Sebastien Boucksom)

On the one hand, we could start with a Calabi–Yau manifold over $\mathbb{C}$, with volume form $\omega \wedge \overline{\omega}$. Or we could start with a Calabi–Yau manifold X over $\mathbb{Q}_p$ and a metrization of K_X, as described in Tschinkel's talk, which produces a measure on $X(\mathbb{Q}_p)$. Passing to finite extensions, we get also measures on $X(L)$ for all finite extensions $L|\mathbb{Q}_p$.

Problem: What is the limiting behavior of these measures supported on $X(L)$, when viewed as measures on the Berkovich space X^{an}, and is there any relation to $\omega \wedge \overline{\omega}$ (after base change from $\mathbb{Q}_p$ to $\mathbb{C}$)?

More generally, it should be interesting to look for nonarchimedean analogues of the bijection in complex geometry between metrics on the canonical bundle K_X and measures on $X(\mathbb{C})$.

12. A quasi-isomorphism from forms to currents. (Klaus Künnemann)

Let X be an analytic space. Philipp Jell has proved a Poincaré Lemma for the complex of forms $(A(X), d')$ arising from the work of Lagerberg and of Chambert-Loir–Ducros, so (at least when X is paracompact and good) the d' cohomology of forms is the cohomology of the underlying topological space of X. We also have natural maps $A(X) \to D(X)$, where $D(X)$ is the complex of currents with respect to d', and one can make similar constructions with d'' and $d' + d''$.

Problem: Are the natural maps $A(X) \to D(X)$ quasi-isomorphisms?

One might also look for a suitable analogue for the δ-forms and δ-currents appearing in the work of Gubler and Künnemann.

13. Green functions and Bernstein's theorems. (Antoine Chambert-Loir)

There is a version of the Bernstein–Kouchnirenko theorem on complete intersections in tori and mixed volumes of polytopes in Lagerberg's thesis.

Problem: Can one use the theory of δ-currents and δ-forms due to Gubler and Künnemann to reprove or generalize results such as the Bernstein–Kouchnirenko Theorem and Rabinoff's Lifting Theorem?

One key step would be to compute the $\star$-product of the δ-currents $[-\log |f_i|]$ on X, for $0 \leq i \leq \dim X$.

14. Minimal Model Program over formal power series rings. (Kiran Kedlaya)

Many of the results discussed at this conference for varieties over $\mathbb{C}[\![t]\!]$ carried an additional hypothesis that the varieties must be defined over the function field of an algebraic curve over $\mathbb{C}$, due to the lack of the full array of vanishing theorems and relative minimal model program over $\mathbb{C}[\![t]\!]$. Recently, Mustaţă and Nicaise proved analogues of relative Kodaira vanishing and Kawamat–Viehweg vanishing over $\mathbb{C}[\![t]\!]$.

Problem: Are these vanishing theorems enough to prove the relative minimal model program over $\mathbb{C}[\![t]\!]$?

There seem to be additional difficulties over a higher dimensional base, e.g. to prove the analogous vanishing theorems over $\mathbb{C}[\![t_1, t_2]\!]$.

15. Log terminal singularities in the non-noetherian setting. (Michael Temkin)

There is a definition of Kawamata log terminal singularities over more general rings (not necessarily noetherian), such as the valuation ring in the algebraic closure of a discretely valued field, involving the space of all possible valuations.

Problem: Do Kawamata log terminal singularities over non-noetherian rings have properties analogous to those in the noetherian setting?

16. Canonical retractions to essential skeletons. (Matt Baker)

By the work of Nicaise–Xu, there is a strong deformation retraction from the Berkovich space of the analytification of a smooth proper variety X over a non-Archimedean field of equal characteristic zero onto its essential skeleton. This comes from an identification of the essential skeleton with the skeleton of the minimal dlt-model of X, and makes essential use of the minimal model program. In particular, since it uses the MMP, the deformation retraction obtained in this way depends on a number of choices.

Problem: Is there a canonical deformation retraction onto the essential skeleton? If not, what data does one need to fix in order to obtain such a retraction map? On a related note, is there a way (canonical or not) to integrate the gradient flow of the weight function in order to obtain an analytic realization of MMP in which one contracts the space X^{an} onto its essential skeleton by following this flow? (This can be thought of as asking for a nonarchimedean analogue of the Kähler-Ricci flow.)

17. Connectedness through codimension 1. (Antoine Chambert-Loir)

The tropicalization of an algebraic subvariety of a torus is connected through codimension 1. It would be useful to have an appropriate analogue in the analytic setting.

Problem: If $U \subset \mathbb{R}^n$ is open and $X \subset \mathrm{Trop}^{-1}(U)$ is an irreducible Zariski closed analytic subvariety, does it follow that $\mathrm{Trop}(X)$ is connected through codimension 1?

As a starting point, one could try to use the balancing formula, interpreted as Stokes's formula in the differential forms framework developed by Chambert-Loir and Ducros to reprove connectedness through codimension 1 in the algebraic case.