$k(U_n(\mathbb{F}_q))$

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$k(U_n\mathbb{F}_q))$

1. Preliminary observations, notation and statement of results.

The aim of this paper is to provide a framework to attack the following conjecture:

(C) For all $n \in \mathbb{N}$, there is a polynomial $f_n(x) \in \mathbb{Z}[x]$ such that

$$k(U_n(\mathbb{F}_q)) = f_n(q) \text{ for all } \mathbb{F}_q.$$

If G is a group, $k(G)$ denotes the cardinality of the set $ccl(G)$ of conjugacy classes of G. If F is a field, then $GL_n(F)$ is a $(B, N)-$pair, and I set

$$U = U_n(F), \quad H = H(F^\times) = \text{ the diagonal matrices in } GL_n(F),$$

$$N = \text{ the monomial matrices in } GL_n(F), \tag{1.1}$$

$$W = N/H, \quad P = \text{ permutation matrices},$$

and I use the isomorphism

$$\begin{aligned} \iota_n = \iota : S_n &\to P \\ \sigma &\mapsto \sum_{i=1}^{n} e_{i,i\sigma} = \sigma\iota. \end{aligned} \tag{1.2}$$

The iota is often suppressed, and when context permits, S_n, P and W are coalesced.

This notation is standard, and when it is helpful for reasons of clarity to specify n,

I write $U_n, H_n, \ldots$ for $U, H, \ldots$

The group $G \times G$ acts on G by the rule

$$
\begin{aligned}
&G \times (G \times G) \to G \\
(1.3) \qquad &(g, (x, y)) \mapsto g \circ (x, y) = x^{-1} g y.
\end{aligned}
$$

If $\Gamma \leq G \times G$, $\quad X$ is a $\Gamma-$set, and $x \in X$, then Γ_x denotes the stabilizer of x in Γ.

Naturally, we make use of the diagonal map

$$
\begin{aligned}
&\delta : G \to G \times G \\
(1.4) \qquad &g \mapsto \delta(g) = (g, g).
\end{aligned}
$$

Let $\mathcal{U}_n(F)$ be the ring of strictly upper triangular $n \times n$ matrices over F, so that $1 + \mathcal{U}_n(F) = U_n(F)$. Since $g^{-1}(1+u)g = 1 + g^{-1}ug, U_n(F)$ and $\mathcal{U}_n(F)$ are isomorphic $\delta(U_n(F))-$sets, and $U_n(F) \times U_n(F)$ stabilizes $\mathcal{U}_n(F)$. Since

$$
(1.5) \qquad \mathcal{U}_n(F) = \bigcup_{d=0}^{n-1} \mathcal{U}_{d,n}(F),
$$

where $\mathcal{U}_{d,n}(F)$ is the set of elements of $\mathcal{U}_n(F)$ of rank d, and since $\mathcal{U}_{0,n}(F) = \{0\}$, we get

$$
(1.6) \qquad k(U_n(\mathbb{F}_q)) = 1 + \sum_{d=1}^{n-1} a_{d,n}(q),
$$

where $a_{d,n}(q)$ is the number of orbits of $\delta(U_n(\mathbb{F}_q))$ in $\mathcal{U}_{d,n}(\mathbb{F}_q)$. It is the $a_{d,n}(q)$ which are studied in this paper. If $f : X \to Y$ is a map, set

$$
(1.7) \qquad \mathcal{D}(f) = X, \quad \mathcal{R}(f) = Y, \quad im(f) = f(X).
$$

If $m, n \in \mathbb{Z}$, set

$$
(1.8) \qquad [m, n] = \{z \in \mathbb{Z} | m \leq z \leq n\}.
$$

Set

$$A_{n-1} = \{(z_1 \ldots, z_n) \in \mathbb{Z}^n | z_1 + z_2 \cdots + z_n = 0\} \tag{1.9}$$

and let $\sum = \sum_{n-1}$ be the corresponding root system:

$$\sum = \{e_i - e_j | i \neq j\} \tag{1.10}$$

where $\{e_i\}$ is the standard basis for $\mathbb{Z}^n$, and by abuse, is the standard basis for R^n for every commutative ring R with 1. The set $\sum^+$ of positive roots is $\{r = e_i - e_j | i < j\}$ and $\sum^- = -\sum^+$. (The positive elements of $\mathbb{R}^n$ are those whose first nonzero coordinate is positive). If $S \subseteq [1, n]^2, R^+(S)$ denotes the set of positive roots $e_i - e_j$ such that $(i, j) \in S$, and $R^-(S)$ denotes the set of negative roots $e_i - e_j$ such that $(i, j) \in S$.

If $R \subseteq \sum$ we say that

$$R \text{ is closed if and only of } (R + R) \cap \sum \subseteq R.$$

When R is a closed, set

$$U_n(R, F) = \langle X_r(F) | r \in R \rangle. \tag{1.11}$$

Here

$$x_r(t) = 1 + te_{ij} \text{ if } r = e_i - e_j \in \sum, \tag{1.12}$$

and

$$X_r(F) = \langle x_r(t) | t \in F \rangle. \tag{1.13}$$

If $R \subseteq \sum^+$, then we adopt the convention that

$$R' \text{ is the complement of } R \text{ in } \sum^+. \tag{1.14}$$

If ω is in S_n, P or W, set

$$R_\omega = \{r \in \sum^+ \mid \quad r\omega < 0\}, \tag{1.15}$$

and let ω_0 be the unique element (of S_n, P, or W, as the case may be) such that

$$R_{\omega_0} = \sum^+. \tag{1.16}$$

Thus, $i\omega_0 = n + 1 - i$ for all $i \in [1, n]$ and if $I \in P_d([1, n])$, then since $[1, n] = I\omega_0 \dot{\cup} I'\omega_0$, it follows that $(I')\omega_0 = (I\omega_0)'$ where $'$ now denotes complementation in $[1, n]$. Set

$$U_{n,\omega}(F) = U_\omega(F) = \langle X_r(F) | r \in R_\omega \rangle.$$

Then we have the basic result that

$$U(F) = U_\omega(F) \cdot U_{\omega\omega_0}(F), U_\omega(F) \cap U_{\omega\omega_0}(F) = 1. \tag{1.18}$$

We next record that

(1.19) if R, R' are complementary sets of roots in $\sum^+$ and both are closed, then there is $\omega \in W$ such that

$$R = R_\omega.$$

This easily proved fact is helpful in this paper.

The set of all $d-$element subsets of the set S is denoted by $P_d(S)$. If $I, J \in P_d([1, n])$, then

(1.20) $\lambda(I, J)$ is the unique order-preserving map from I to J, (1.21) $\lambda^-(I,J)$ is the unique order-reversing map from I to J,

with the convention that $\lambda^-(I,J) = \lambda(I,J)$ if $d=1$. Also $\pi(I)$ is defined:

$$\begin{aligned}&\pi(I) \text{ is the unique element of } S_n \text{ which agrees with } \lambda(I,[1,d]) \text{ on } I \text{ and}\\ &\text{agrees with } \lambda(I',[d+1,n]) \text{ on } I'.\end{aligned} \tag{1.22}$$

And $\rho(I)$ is defined as the unique element of S_n which agrees with

$\lambda^-(I,[n-d+1,n])$ on I and agrees with $\lambda^-(I',[1,n-d])$ on I'. A picture of $\rho(I)$ might look something like

$$\begin{array}{ccc} I & & I' \\ o\times\times\times & & \times o\times\times \\ & \rho(I) & \\ \times\times o\times & & \times\times\times o \\ [1,n-d] & & [n-d+1,n] \end{array} \tag{1.23}$$

We check directly that

$$U(R^+(I\times I'),F) = U_{\pi(I')}(F), \tag{1.24}$$

with analogous equalities for other subsets $[1,n]^2$ which have suitable box-like properties.

We embed S_d in S_n in the usual way by extending each σ in S_d to the element of S_n which agrees with σ on $[1,d]$ and fixes every element of $[d+1,n]$. Similarly, we embed $GL_d(F)$ in $GL_n(F)$, sending g in $GL_d(F)$ to

$$\begin{pmatrix} g & 0 \\ 0 & 1_{n-d}\end{pmatrix}.$$

If $I,J\in P_d[1,n]$, set

$$\begin{aligned}C_n(I,J) = \{\omega\in S_d | i < i\pi(I)\omega\pi(J)^{-1} \\ \text{for all } i\in I\}\end{aligned} \tag{1.25}$$

(1.26) $$P_n(I,J)=\{\omega\iota|\omega\in C_n(I,J)\}.$$

$C_n(I,J)$ is often empty, but we study carefully the set of triples (I,J,ω), where $I,J\in P_d([1,n])$ and $\omega\in C_n(I,J)$. Suppose (I,J,ω) is such a triple. Write

(1.27) $$I=\{i_1,\dots,i_d\} \qquad J=\{j_1,\dots,j_d\}$$

$i_1<i_2<\cdots<i_d$, $\quad j_1<j_2<\cdots<j_d$. Hence, for $m\in[1,d]$, $\quad i_m\pi(I)=m, m\pi(J)^{-1}=j_m$, and so $i_m\pi(I)\omega\pi(J)^{-1}=(m\omega)\pi(J)^{-1}=j_{m\omega}$ and so (1.25) yields

(1.28) $$i_m<j_{m\omega},\quad m\in[1,d].$$

I continue to examine (I,J,ω). Since $[1,n]=I\dot\cup I'$, we have

$$J=I\cap J\dot\cup I'\cap J,$$

and similarly,

$$I=I\cap J\dot\cup I'\cap J,$$

So $d=|J|=|I\cap J|+|I'\cap J|, d=|I|=|I\cap J|+|I\cap J'|$, so

$$|I'\cap J|=|I\cap J'|.$$

Since $i_d<j_{d\omega}\le j_d$, we get $j_d\notin I$, and so $I\ne J$. Since $|I|=|J|$, this forces

(1.29) $$I'\cap J\ne\phi,\quad I\cap J'\ne\phi.$$

Let $\Lambda = \Lambda(I, J)$ be the set of all maps λ with $\mathcal{D}(\lambda) \subseteq I' \cap J, \mathcal{R}(\lambda) = I \cap J'$, such that

$$(1.30) \qquad \begin{array}{ll} (i) & \lambda : \mathcal{D}(\lambda) \to I \cap J' \text{ is an injection.} \\ (ii) & a < a\lambda \text{ for all } a \in \mathcal{D}(\lambda). \end{array}$$

I adopt the convention that the empty map is in Λ. This is the map λ with $\mathcal{D}(\lambda) = \phi, \quad \mathcal{R}(\lambda) = I \cap J'$. If, for example, $i_d < j_1$, then Λ consists only of the empty map. We shall, however, meet some large Λ.

It is extremely helpful to observe that

$$(1.31) \qquad \mathcal{D}(\lambda) \subset I' \cap J \text{ for all } \lambda \in \Lambda.$$

For if $\mathcal{D}(\lambda) = I' \cap J$, then λ is a bijection between $I' \cap J$ and $I \cap J'$, and since λ is increasing by (1.30. ii), we get

$$\sum_{a \in I' \cap J} a < \sum_{b \in I \cap J'} b,$$

and so

$$\sum_{a \in J} a < \sum_{b \in I} b.$$

But

$$\sum_{b \in I} b = \sum_{m \in [1,d]} i_m < \sum_{m \in [1,d]} j_{m\omega} = \sum_{m \in [1,d]} j_m = \sum_{a \in J} a.$$

So (1.31) holds.

We next consider a 4-tuple (I, J, ω, λ), where $\omega \in C_n(I, J), \lambda \in \Lambda(I, J)$. For each field F, I define a subgroup $\Gamma = \Gamma(I, J, \lambda, F)$ of $U_d(F) \times U_d(F)$, by giving a set $\mathcal{G}$ of generators. Set

$$(1.32) \qquad \begin{array}{ll} A = \mathcal{D}(\lambda), & C = I' \cap J \backslash A, \\ B = \mathcal{D}(\lambda)\lambda, & D = I \cap J' \backslash B, \\ \pi_1 = \pi(I), & \pi_2 = \pi(J). \end{array}$$

We take $\mathcal{G}$ to be the set of displayed elements.

$$(U_d(R^+(D\pi_1 \times [1,d]),F),1). \tag{1.33}$$

$$(1,U_d(R^+([1,d] \times C\pi_2),F)). \tag{1.34}$$

$$\begin{gathered}(x_{i\pi_1,j\pi_1}(t), x_{i\pi_2,j\pi_2}(t)), t \in F,\\ e_i - e_j \in R^+(I \cap J \times I \cap J).\end{gathered} \tag{1.35}$$

$$\begin{gathered}(x_{a\lambda\pi_1,a'\lambda\pi_1}(t), x_{a\pi_2,a'\pi_2}(t)), t \in F,\\ a, a' \in A, a < a', a\lambda < a'\lambda.\end{gathered} \tag{1.36}$$

$$\begin{gathered}(x_{a\lambda\pi_1,j\pi_1}(t), x_{a\pi_2,j\pi_2}(t)), t \in F, a \in A,\\ j \in I \cap J, a\lambda < j.\end{gathered} \tag{1.37}$$

$$\begin{gathered}(x_{j\pi_1,a\lambda\pi_1}(t), \quad x_{j\pi_2,a\pi_2}(t)), t \in F,\\ j \in I \cap J, a \in A, j < a.\end{gathered} \tag{1.38}$$

This gives us $\Gamma(I,J,\lambda,F)$. Let

$$\begin{gathered}f(I,J,\omega,\lambda,q) \text{ be the number of orbits of}\\ \Gamma(I,J,\lambda,\mathbb{F}_q) \text{ on } U_d(\mathbb{F}_q)\omega\iota U_d(\mathbb{F}_q).\end{gathered} \tag{1.39}$$

The appearance of ι in (1.39) makes clear that we are to examine $U_d(\mathbb{F}_q)P_dU_d(\mathbb{F}_q)$.

The Cartan subgroup has disappeared. It is to be understood that $\Gamma \leq GL_d(F) \times GL_d(F)$ and that $U_d(F)\omega\iota U_d(F) \subseteq GL_d(F)$, so that (1.3) applies.

Theorem 1. *If $1 \leq d \leq n-1$, then*

$$a_{d,n}(q) = \sum (q-1)^{d+|\mathcal{D}(\lambda)|} \cdot f(I,J,\omega,\lambda,q),$$

where the sum is over all 4-tuples (I,J,ω,λ) such that $I,J \in P_d([1,n]), \omega \in C_n(I,J), \lambda \in \Lambda(I,J)$.

Theorem 2. *If $I, J \in P_d([1,n])$ and $\lambda \in \Lambda(I,J)$, then there are $\sigma_1, \sigma_2, \tau_1, \tau_2 \in S_d$ such that for all fields F, there are exact sequences*

$$1 \to U_{\tau_1}(F) \to \Gamma(I,J,\lambda,F) \stackrel{p_1}{\to} U_{\sigma_1}(F) \to 1,$$

$$1 \to U_{\tau_2}(F) \to \Gamma(I,J,\lambda,F) \stackrel{p_2}{\to} U_{\sigma_2}(F) \to 1,$$

where p_i is the projection of Γ to the i^{th} factor of $U_d(F) \times U_d(F)$.

One of the building blocks in the proofs of Theorems 1 and 2 is of independent interest.

Theorem 3. *Suppose $I, J \in P_d([1,n])$. For each non empty $\lambda \in \Lambda(I,J)$, and each field F, set*

$$T(\lambda,F) = \left\{ \prod_{a \in \mathcal{D}(\lambda)} x_{a,a\lambda}(t_a) | t_a \in F^{\times} \right\},$$
$$T(I,J,F) = \{1\} \cup \bigcup T(\lambda, F),$$

where the union is over the non empty $\lambda \in \Lambda(I,J)$. Then

$$U_n(F) = \dot{\bigcup_{g \in T(I,J,F)}} U_{\rho(J')}(F) g U_{\rho(I)}(F).$$

Theorem 3 gives an explicit description of the double coset space

$$U_{\rho(J')}(F) \backslash U_n(F) / U_{\rho(I)}(F),$$

for all pairs of $d-$element subsets of $[1,n]$ and fields F. This space is the disjoint union of copies of $F^{\times^h}$, where h ranges over a multi set of non negative integers, precisely one of which is zero.

The stabilizer of E in $G \times G$.

Set $G = GL_n(F), \Gamma = G \times G, \quad E_d = E = \sum_{i=1}^{d} e_{ii}$, where $1 \leq d \leq n-1$, and F is a field. Denote by $M_n(F)$ the set of $n \times n$ matrices over F. The group Γ acts on $M_n(F)$, and so Γ acts on ${}_dM_n(F)$, the set of elements of $M_n(F)$ of rank d. The action is the usual one

$$M_n(F) \times \Gamma \to M_n(F)$$

$$(M, (g, g')) \mapsto M \circ (g, g') = g^{-1} M g'.$$

Since Γ acts transitively on ${}_dM_n(F)$, and since $E \in {}_dM_n(F)$, every element of ${}_dM_n(F)$ is of the form $P^{-1}EQ$, where $(P, Q) \in \Gamma$. The representation is not unique. I propose to remedy this.

We examine $(g, g') \in \Gamma_E$. Write

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad g' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix},$$

where $\alpha, \alpha' \in M_d(F)$. We have

$$\begin{pmatrix} \alpha & 0 \\ \gamma & 0 \end{pmatrix} = gE = eg' = \begin{pmatrix} \alpha' & \beta' \\ 0 & 0 \end{pmatrix}.$$

so

$$\alpha = \alpha', \gamma = 0, \beta' = 0.$$

Since g and g' are non singular, we get that $\alpha \in GL_d(F)$,

$$g = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}, g' = \begin{pmatrix} \alpha & 0 \\ \gamma' & \delta' \end{pmatrix}.$$

Set

$$P_0^d = P_0 = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \middle| \alpha \in GL_d(F), \delta \in GL_{n-d}(F), \right.$$
$$\left. \beta \in M_{d,n-d}(F) \right\},$$
$$P_d^0 = P^0 = \left\{ \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \middle| \alpha \in GL_d(F), \delta \in GL_{n-d}(F), \right.$$
$$\left. \gamma \in M_{n-d,d}(F) \right\}$$

Thus,

$$\Gamma_E = \{(g, g') \in P_0 \times P^0 | EgE = Eg'E\}.$$

3. The coset structure of P_0 in G.

We study the action of $G \times P_0$ on $G \times G$. I construct two sets of representatives for the set $\{gP_0 | g \in G\}$ of cosets of P_0 in G. I call them T_0 and T_0'. I begin with

$$M = (\alpha_{i,j}) \in G \quad (G = GL_n(F)).$$

For $i \in [1, n]$, set

$$v_i = (\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{id}) \in F^d.$$

Set $V_0 = 0, V_i = \sum_{j=1}^{i} Fv_j$. This gives us a chain $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n$ of subspaces of F^d. Set $r_i = \dim V_i$, so that

$$0 = r_0 \leq r_1 \leq \cdots \leq r_n = d,$$

the equality holding since row and column rank coincide, and since M is non singular, so that its columns are linearly independent.

Since $V_i = V_{i-1} + Fv_i$, we get

$$r_{i-1} \leq r_i \leq r_{i-1} + 1, \quad i \in [1, n].$$

Since $d = r_n = \sum_{i=0}^{n-1} (r_{i+1} - r_i)$, there is $I \in P_d([1, n])$ such that

$$r_i = \begin{cases} r_{i-1} + 1 & \text{if } i \in I. \\ r_{i-1} & \text{if } i \in I'. \end{cases}$$

From the construction of the v_j, we get that for each $j \in I'$, there are $c_{ij} \in F, i \in I, i < j$, such that

$$v_j = \sum c_{ij} v_i. \tag{3.1}$$

Set $I = I(M)$. We check that $I(M) = I(Mg)$ for all $g \in P_0$. This equality obviously holds if $g \in H_n(F^\times)$, while if $g = x_{\alpha\beta}(t)$, and $(\alpha, \beta) \in ([1, d] \times [1, n]) \cup ([d+1, n] \times [d+1, n])$, the equality is easily checked, using once again that row rank and column rank coincide. Indeed, if $Mx_{\alpha\beta}(t) = M' = (a'_{ij})$, then for all $i \in [1, n]$, the matrices

$$\begin{pmatrix} a_{11} & \dots & a_{1d} \\ \vdots & & \\ a_{i1} & \dots & a_{id} \end{pmatrix}, \begin{pmatrix} a'_{11} & \dots & a'_{1d} \\ \vdots & & \\ a'_{i1} & \dots & a'_{id} \end{pmatrix}$$

have the same column rank. Since $H_n(F^\times)$ and $\{x_{\alpha\beta}(t) | t \in F, (\alpha, \beta) \in ([1, d] \times [1, n]) \cup ([d + 1, n] \times [d + 1, n])\}$ generate $P_0, I(M)$ is constant on MP_0.

Set $x(M) = \prod x_{ji}(-c_{ij})$, where $j \in I', i \in I, i < j$, and where the c_{ij} are given in (3.1). The order of the product is immaterial, since $U(R^-(I' \times I), F)$ is abelian.

Set $\tilde{M} = x(M)M = (\tilde{a}_{ij})$. Thus,

$$\tilde{a}_{ij} = 0 \text{ for all } i \in I', j \in [1, d].$$

This tells us that $\pi(I)^{-1}\tilde{M} \in P_0$. It would be more accurate to write $(\pi(I)\iota)^{-1}$ in place of $\pi(I)^{-1}$, but by abuse, I omit the iota. So

$$M \in U(R^-(I' \times I), F)\pi(I)P_0. \tag{3.2}$$

Set

$$T'_0 = \bigcup_{I \in P_d([1,n])} U(R^-(I' \times I), F)\pi(I). \tag{3.3}$$

Since $I(M) = I(Mg)$ for all $M \in G, f \in P_0$, it is straightforward to check that

$$T'_0 P_0 = G, T_0'^{-1} T'_0 \cap P_0 = \{1\}.$$

Set

$$T^0 =^t T_0',$$

the set of transposes of elements of T_0'. Since ${}^tP_0 = P^0$, and since ${}^t\pi(I) = \pi(I)^{-1}$, and since ${}^tU(R^-(I' \times I), F) = U(R^+(I \times I'), F)$, it follows that

$$T^0 = \bigcup_{I \in P_d([1,n])} \pi(I)^{-1} U(R^+(I \times I'), F), \tag{3.4}$$

$$P^0 T^0 = G, \quad P^0 \cap T^0 T^{0-1} = \{1\}. \tag{3.5}$$

We start again. Set $W_0 = 0, W_1 = F\upsilon_n$,

$$W_j = \sum_{k=n-j+1}^{k=n} F\upsilon_k.$$

Then $W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n$, and if we set $s_i = \dim W_i$, then $0 = s_0 \leq s_1 \leq \cdots \leq s_n = d$. So there is $J \in P_d(\underline{n})$ such that

$$s_j - s_{j-1} = \begin{cases} 1 & \text{if } j \in J. \\ 0 & \text{if } j \in J'. \end{cases}$$

Thus, we get, for $j \in J'$,

$$\upsilon_j = \sum c'_{jk} \upsilon_k$$

where $c'_{jk} \in F$, and the sum is over $k \in I, k > j$. Set

$$\tilde{M} = xM = (\tilde{a}_{ij}),$$

where $x = x(M) = \prod x_{jk}(-c'_{jk})$, where the sum is over $j \in I', k \in I, j < k$. Thus

$$\tilde{a}_{ij} = 0 \text{ for all } i \in J', j \in [1, d].$$

Hence, $\pi(J)^{-1}\tilde{M} \in P_0$, or equivalently,

$$M \in x(M)^{-1}\pi(J)P_0.$$

Set

$$T_0 = \bigcup_{j \in P_d[1,n]} U(R^+(J' \times J), F)\pi(J).$$

Then we have shown that

$$(3.6) \qquad T_0 P_0 = G, \quad T_0^{-1} T_0 \cap P_0 = \{1\}.$$

Putting these pieces together, we conclude that

(3.7) Every $X \in {}_dM_n(F)$ has a representation as

$$X = t_0 E g E t^0, t_0 \in T_0, t^0 \in T^0, g \in GL_d(F).$$

In addition, we get that if $X = t_0' E g' E t^{0\prime}$, where $t_0' \in T_0, t^{0\prime}, g' \in GL_d(F)$, then $t_0 = t_0', t^0 = t^{0\prime}, g = g'$. I call (3.7) the normal form of X. Note also that if $t_0 \in T_0$, then $t_0 = x_1\pi(I_1), x_1 \in U(R^+(I_1' \times I_1), F), I_1 \in P_d[1,n]$, and if $t_0 = x_2\pi(I_2)$, where $x_2 \in U(R^+(I_2' \times I_2), F), I_2 \in P_d([1,n])$, then $x_1 = x_2, I_1 = I_2$, and similarly for t^0. Thus, if $X \in {}_dM_n(F)$, and

$$X = x\pi(I)EgE\pi(J)^{-1}y,$$

where $x \in U(R^+(I' \times I), F), y \in U(R^+(J \times J'), F), g \in GL_d(F)$, then the 5-tuple (x, I, g, J, y) is uniquely determined by X.

If $g \in GL_d(F)$, then

$$g = u_1 h \omega u_2, u_1, u_2 \in U_d(F), h \in H_d(F^\times), \omega \in P_d,$$

and so

$$X = x\pi(I)Eu_1h\omega u_2E\pi(J)^{-1}y.$$

Note that

$$\pi(I)EU_d(F) = U_n(R^+(I \times I), F)\pi(I)E,$$

and

$$U_d(F)E\pi(J)^{-1} = E\pi(J)^{-1}U_n(R^+(J \times J), F).$$

Thus

$$X \in U_n(F)\pi(I)Eh\omega E\pi(J)^{-1}U_n(F),$$

whence

$$X \in \mathcal{U}_{d,n}(F) \Leftrightarrow \pi(I)Eh\omega E\pi(J)^{-1} \in \mathcal{U}_{d,n}(F).$$

If we note that for all $i \in I'$,

$$e_{i\pi(I)}E = 0,$$

we conclude that

$$X \in \mathcal{U}_{d,n}(F) \Leftrightarrow i < i\pi(I)\omega\pi(J)^{-1} \text{ for all } i \in I. \tag{3.8}$$

By (1.25), we conclude that

$$\mathcal{U}_{d,n}(F) = \bigcup U_n(F)\pi(I)Eh\omega\iota E\pi(J)^{-1}U_n(F), \tag{3.9}$$

where the union is over all 4-tuples $(I, J, \omega, h), I, J \in P_d([1, n]), \omega \in C_n(I, J), h \in H_d(F^\times)$.

4. (I, J, ω).

I examine the process whereby an element of

$$U_n(F)\pi(I)Eh\omega E\pi(J)^{-1}U_n(F)$$

is put in normal form. If $u \in U_n(F)$, then by (1.18) and (1.24),

$$u = x\tilde{u}, x \in U(R^+(I' \times I), F),$$

$\tilde{u} \in U(R^+(I \times I) \cup R^+([1,n] \times I'), F)$, so $\tilde{u} = tz, t \in U(R^+(I \times I), F), z \in U(R^+(I \times I'), F)$. Then $z\pi(I)E = \pi(I) \cdot x^{\pi(I)}E = \pi(I)E$, whence $u\pi(I)E = xt\pi(I)E = x\pi(I)Et^{\pi(I)}$, with $t^{\pi(I)} \in U_d(F)$. A similar argument with $E\pi(J)^{-1}U_n(F)$ leads to the normal form of the element being examined. There are 4 relevant subgroups of $U_n(F)$ which are involved in this process:

$$U(R^+(I' \times I), F), U(R^+(I \times I) \cup R^+([1,n] \times I'), F) \text{ for } \pi(I),$$

$$U(R^+(J' \times [1,n]) \cup R^+(J \times J), F), U(R^+(J \times J'), F) \text{ for } \pi(J)^{-1}.$$

Since

$$R^+(I \times I) \cup R^+([1,n] \times I') = R_{\rho(I)},$$

and

$$R^+(J' \times [1,n]) \cup R^+(J \times J) = R_{\rho(J')},$$

the $(U_{\rho(J')}(F), U_{\rho(I)}(F))$ double cosets in $U_n(F)$ begin to emerge.

Pick $u, u' \in U_n(F)$ and consider the $\delta(U_n(F))$ orbit O which contains

$$u\pi(I)Eh\omega E\pi(J)^{-1}u'.$$

Let

$$O' = \{X \in O| \text{ the normal form for } X \text{ is}$$

$$1 \cdot \pi(I)EgE\pi(J)^{-1}y, y \in U_{\rho(J')\omega_o}(F),$$

$$g \in U_d(F)h\omega U_d(F)\}.$$

Note that $U_{\rho(J')\omega_0}(F)$ is just another name for $U(R^+(J \times J'), F)$. Obviously, $O' \neq \phi$, since

$$u\pi(I)Eh\omega E\pi(J)^{-1}u' \circ \delta(u) \in O'.$$

Next we observe that if $Y \in O', x \in U_n(F)$ and $Y \circ \delta(x) \in O'$, then $x^{-1} \in U_{\rho(I)}(F)$, in which case $O' = O' \circ \delta(x)$. So

$$\delta(U_{\rho(I)}(F)) \text{ is the stabilizer of } O' \text{ in}$$

$$\delta(U_n(F)),$$

and O' is a $\delta(U_{\rho(I)}(F))-$orbit. Let

$$\mathcal{L}(I, J, h, w) = \{w \in U_n(F)\pi(I)Eh\omega E\pi(J)^{-1}U_n(F)|$$

$$\text{the normal form for } w \text{ is}$$

$$1 \cdot \pi(I)EgE\pi(J)^{-1}y, y \in U_{\rho(J')\omega_0}(F),$$

$$g \in U_d(F)h\omega U_d(F).\}.$$

We have just shown that there is a bijection between the $\delta(U_n(F))$ orbits on $U_n(F)\pi(I)Eh\omega E\pi(J)^{-1}U_n(F)$ and the $\delta(U_{\rho(I)}(F))$ orbits on $\mathcal{L}(I, J, h, \omega)$. Note that since $Eg = gE = EgE$ for all $g \in GL_d(F)$, we can dispense with one of the E's in EgE, and write Eg. Next, we prove that

$$\text{if } g, g' \in GL_d(F), y, y' \in U_{\rho(J')\omega_o}(F) \text{ and}$$

$$\pi(I)Eg\pi(J)^{-1}y \text{ and } \pi(I)Eg'\pi(J)^{-1}y'$$

are in the same $\delta(U_{\rho(I)}(F))-$orbit, then

$$U_{\rho(J')}(F)yU_{\rho(I)}(F) = U_{\rho(J')}(F)y'U_{\rho(I)}(F).$$

For suppose that

$$\pi(I)Eg'\pi(J)^{-1}y' = \pi(I)Eg\pi(J)^{-1}y \circ \delta(u),$$

where $u \in U_{\rho(I)}(F)$. Write $u^{-1} = w \cdot w'$, where $w' \in U(R^+([1,n] \times I'), F), w \in U(R^+(I \times I), F)$, set $u_1 = w^{\pi(I)} \in U_d(F)$, and get

$$\pi(I)Eg'\pi(J)^{-1}y' = \pi(I)Eu_1g\pi(J)^{-1}yu.$$

Write $yu = zy_1$, where $z \in U_{\rho(J')}(F), y_1 \in U_{\rho(J')\omega_0}(F)$. Then write $z = z_2z_1, z_2 \in U(R^+(J' \times [1,n]), F), z_1 \in U(R^+(J \times J), F)$, set $u_2 = z_1^{\pi(J)} \in U_d(F)$, and get

$$\begin{aligned}\pi(I)Eu_1g\pi(J)^{-1}yu &= \pi(I)u_1gE\pi(J)^{-1}z_2z_1y_1 \\ &= \pi(I)u_1gE\pi(J)^{-1}z_1y_1 \\ &= \pi(I)u_1gu_2E\pi(J)^{-1}y_1\end{aligned}$$

and so by uniqueness of the normal form,

$$g' = u_1gu_2, \quad y' = y_1.$$

Hence $y' = y_1 = z^{-1}yu \in U_{\rho(J')}(F)yU_{\rho(I)}(F)$, and our assertion is proved.

To continue the discussion, I assume for the remainder of this section that Theorem 3 is available. By that theorem and by what we have just shown there is $T \in T(I, J, F)$ such that every element $\pi(I)Eg\pi(J)^{-1}y$ of O', with $g \in GL_d(F), y \in U_{\rho(J')\omega_0}(F)$, has the property that $y \in U_{\rho(J')}(F)TU_{\rho(I)}(F)$. We now observe that $T(I, J, F) \subseteq U_{\rho(J')\omega_0}(F)$, a remark which could have been made earlier and is hardly surprising, but nevertheless needs to be mentioned since it means that $\pi(I)Eg\pi(J)^{-1}T$ is in normal form for all $g \in GL_d(F), T \in T(I, J, F)$. So we are led to

$$\mathcal{U}_{d,n}(I, J, h, \omega, T, F) =$$

$$\{Z \in \mathcal{U}_{d,n}(F)|$$

$$Z = \pi(I)Eg\pi(J)^{-1}T, \quad g \in U_d(F)h\omega U_d(F)\}.$$

We need to decide when two elements of this set are in the same $\delta(U_n(F))$−orbit, since Theorem 3 tells us that every orbit of $\delta(U_n(F))$ on $\mathcal{U}_{d,n}(F)$ has a nonempty intersection with $\mathcal{U}_{d,n}(I, J, h, \omega, T, F)$ for a uniquely determined 5-tuple (I, J, h, ω, T) where $I, J \in P_d([1, n]), \omega \in C_n(I, J), h \in H_d(F^\times), T \in T(I, J, F)$.

Suppose $\pi(I)Eg_1\pi(J)^{-1}T, \pi(I)Eg_2\pi(J)^{-1}T \in \mathcal{U}_{d,n}(I, J, h, \omega, T, F)$ and $u \in U_n(F)$ satisfy

$$\pi(I)Eg_2\pi(J)^{-1}T = \pi(I)Eg_1\pi(J)^{-1}T \circ \delta(u).$$

As we have already seen, this forces $u \in U_{\rho(I)}(F)$, which in turn guarantees that

$$u^{-1}\pi(I)Eg_1 = \pi(I)Eu_1g_1$$

for some $u_1 \in U_d(F)$. So we get

$$\pi(I)Eg_2\pi(J)^{-1}T = \pi(I)Eu_1g_1\pi(J)^{-1}Tu.$$

This in turn forces

$$Tu = vT, \quad v \in U_{\rho(J')}(F),$$

and so $(v, u) \in (U_{\rho(J')}(F) \times U_{\rho(I)}(F))_T$. Thus

$$\pi(I)Eg_2\pi(J)^{-1}T =$$

$$\pi(I)Eu_1g_1\pi(J)^{-1}vT = \pi(I)Eu_1g_1u_2\pi(J)^{-1}T$$

for some $u_2 \in U_d(F)$. So $g_2 = u_1g_1u_2$. Conversely, if $(v, u) \in (U_{\rho(J')}(F) \times U_{\rho(I)}(F))_T$ and u_1, u_2 are defined by

$$u^{-1}\pi(I)E = \pi(I)Eu_1,$$

$$E\pi(J)^{-1}v = u_2E\pi(J)^{-1},$$

then $\pi(I)Eg_1\pi(J)^{-1}T$ and $\pi(I)Eu_1g_1u_2\pi(J)^{-1}T$ are in the same $\delta(U_n(F))-$orbit.

It remains to identify (u_1, u_2) from

$$(v, u) \in (U_{\rho(J')}(F) \times U_{\rho(I)}(F))_T.$$

We need to find generators for $(U_{\rho(J')}(F) \times U_{\rho(I)}(F))_T$. Having done so, we need to examine closely the process which converts (v, u) to (u_1, u_2). We have

$$T = \prod_{a \in \mathcal{D}(\lambda)} x_{a,a\lambda}(t_a),$$

where $\lambda \in \Lambda(I, J)$ and each $t_a \in F^\times$. As $h \in H_d(F^\times)$, there are elements $\xi_1, \ldots \xi_d \in F^\times$ such that

$$h = diag(\xi_1, \ldots, \xi_d) = h(\xi_1, \ldots, \xi_d).$$

Set

$$\begin{aligned} &A = \mathcal{D}(\lambda), \quad C = I' \cap J \backslash A, \\ (4.1) \qquad & \\ &B = \mathcal{D}(\lambda)\lambda, \quad D = I \cap J' \backslash B, \quad m = |A|. \end{aligned}$$

In case $m = 0$ so that $T = 1$ and λ is the empty map, some of the following discussion is not needed, but I carry the argument out for all $T \in T(I, J, F)$.

Let $A = \{a_1 \ldots, a_m\}, a_1 < a_2 < \cdots < a_m$. If $\xi_1, \ldots, \xi_d, t_{a_1}, \ldots, t_{a_m} \in F^\times$, set

$$\mathcal{X}(I, J, \omega, \lambda, \xi_1, \ldots, \xi_d, t_{a_1}, \ldots, t_{a_m}, F) =$$

$$\pi(I) E U_d(F) h(\xi_1, \ldots, \xi_d) \omega U_d(F) \pi(J)^{-1} \cdot \prod_{i=1}^{m} x_{a_i a_i \lambda}(t_{a_i}).$$

Set

$$(U_{\rho(J')}(F) \times U_{\rho(I)}(F))_x =$$

$$Q(I, J, \lambda, t_{a_1}, \ldots, t_{a_m}), \text{ where}$$

$$x = \prod_{i=1}^{m} x_{a_i a_i \lambda}(t_{a_i})$$

Set

$$\mathcal{X}(I, J, \omega, \lambda, F) =$$

$$\bigcup_{(F^\times)^{d+m}} \mathcal{X}(I, J, \omega, \lambda, \xi_1, \ldots, \xi_d, t_{a_1}, \ldots, t_{a_m}, F).$$

5. The action of $H_n(F^\times)$ on $\mathcal{X}(I, J, \omega, \lambda, F)$.

Fix $i \in [1, n]$ and for each $c \in F^\times$, set

$$h_i(c) = \sum_{\substack{j=1 \\ j \neq i}}^{n} e_{jj} + c e_{ii},$$

the diagonal matrix with c in position i, 1 elsewhere. If $\eta_1, \ldots, \eta_n \in F^\times$, set $h(\eta_1, \ldots, \eta_n) = \sum_{i=1}^{n} \eta_i e_{ii}$. Pick $X \in \chi(I, J, \omega, \lambda, F)$, and write

$$X = \pi(I) E u_1 h(\xi_1, \ldots, \xi_d) \omega u_2 \pi(J)^{-1} \cdot \prod_{j=1}^{m} x_{a_j, a_j\lambda}(t_{a_j}).$$

We examine closely $Y = h_i(c)^{-1} X h_i(c)$. We have

$$h_i(c)^{-1} \pi(I) = \pi(I) h_{i\pi(I)}(c)^{-1}.$$

For $j \in [1, m]$

$$x_{a_j, a_j\lambda}(t_{a_j}) h_i(c) =$$

$$h_i(c) x_{a_j, a_j\lambda}(c^f t_{a_j}),$$

where

$$f = f(i, a_j) = -\delta_{i, a_j} + \delta_{i, a_j\lambda},$$

and where δ is that of Kronecker.

Hence

$$\pi(J)^{-1} \prod_{j=1}^{m} x_{a_j, a_{j\lambda}}(t_{a_j}) h_i(c) =$$

$$h_{i\pi(J)}(c) \pi(J)^{-1} \cdot \prod_{j=1}^{m} x_{a_j, a_j\lambda}(c^{f(i, a_j)} t_{a_j}).$$

Case 1. $i \in I', i \in J'$.

Here

$$h_{i\pi(I)}(c)^{-1} E = E$$

$$E = Eh_{i\pi(J)}(c).$$

Case 2. $i \in I', i \in J$.

Here

$$h_{i\pi(I)}(c)^{-1}E = E,$$

$$h_{i\pi(J)}(c)E = Eh_{i\pi(J)}(c), \quad i\pi(J) \in [1, d].$$

Case 3. $i \in I, i \in J'$.

Here

$$h_{i\pi(I)}(c)^{-1}E = Eh_{i\pi(I)}(c)^{-1}, \quad i\pi(I) \in [1, d]$$

$$E = Eh_{i\pi(J)}(c).$$

Case 4. $i \in I, i \in J$. Here

$$h_{i\pi(I)}(c)^{-1}E = Eh_{i\pi(I)}(c)^{-1}, \quad i\pi(I) \in [1, d],$$

$$h_{i\pi(J)}(c)E = Eh_{i\pi(J)}(c), \quad i\pi(J) \in [1, d].$$

This tells us that

$$Y = \pi(I)Eu_1'h(\xi_1', \dots, \xi_d')\omega u_2'\pi(J)^{-1}.$$

$$\prod_{j=1}^{m} x_{a_j, a_j\lambda}(t_{a_j}'),$$

where $u_1', u_2' \in U_d(F)$,

$$t_{a_j}' = c^{f(i,a_j)}t_{a_j}, \quad j \in [1, m],$$

$$\xi_k' = c^{g(i,k)}\xi_k, \quad k \in [1, d],$$

where

$$g(i,k) = -1 \text{ if } k = i\pi(I) \text{ and } i \in I,$$

$$= 1 \text{ if } k = i\pi(J)\omega^{-1} \text{ and } i \in J$$

$$= 0 \text{ if } k \neq \{i\pi(I), i\pi(J)\omega^{-1}\}.$$

Here I am using crucially the fact that $i\pi(I) \neq i\pi(J)\omega^{-1}$ for all $i \in I \cap J$, which is a consequence of $\omega \in P_n(I,J)$.

Note too that since $\mathcal{D}(\lambda) \cap \mathcal{D}(\lambda)\lambda = \phi$, it follows that $f(i,a_j) \in \{0,1,-1\}$, and for each $i \in [1,n], \{j \in [1,n] | f(i,a_j) \neq 0\}$ is either empty or has precisely one element. We build a matrix M, indexed by $[1,n] \times [1,d+m]$ whose (i,ℓ) entry is $m_{i\ell}$, and

$$m_{il} = g(i,\ell) \text{ if } \ell \in [1,d],$$

while

$$m_{id+l} = f(i,a_l), \text{ if } d+l \in [d+1,d+m].$$

I aim to prove

$$\text{(5.1)} \qquad \begin{array}{l} (i) \quad M \text{ has } \mathbb{Q}-\text{rank } d+m. \\ (ii) \text{ All } \mathbb{Z}-\text{elementary divisors of } M \text{ are } 1. \end{array}$$

Before tackling this task, note that since $H_n(F^\times)$ normalizes $U_n(F), U_{\rho(J')}(F)$ and $U_{\rho(I)}(F)$, we get

$$h_i(c)^{-1}\Big((U_{\rho(J')}(F) \times U_{\rho(I)}(F))_{\prod_{j=1}^{m} x_{a_j,a_j\lambda(t_{a_j})}}\Big)h_i(c) =$$
$$(U_{\rho(J')}(F) \times U_{\rho(I)}(F))_{\prod_{j=1}^{m} x_{a_j,a_j\lambda(c^{f(i,a_j)})}}$$

that is, $Q(I,J,\lambda,t_{a_1},\ldots,t_{a_m})^{h_i(c)} = Q(I,J,\lambda,t'_{a_1},\ldots,t'_{a_m})$, where $t'_{a_j} = c^{f(i,a_j)}t_{a_j}$, $j \in [1,m]$. Also, since $H_d(F^\times)$ normalizes $U_d(F)$ and ω normalizes $H_d(F^\times)$, the orbits of $Q(I,J,\lambda,t_{a_1},\ldots,t_{a_m})$ on $\mathcal{X}(I,J,\omega,\lambda,\xi_1,\ldots,\xi_d,t_{a_1},\ldots,t_{a_d},F)$ and the orbits of $Q(I,J,\lambda,t'_{a_1},\ldots,t'_{a_m})$ on $\mathcal{X}(I,J,\omega,\lambda,\xi'_1,\ldots,\xi'_d,t'_{a_1},\ldots,t'_{a_m},F)$ (where $\xi'_j = c^{g(i,j)}\xi_j, j \in [1,d]$, are in 1-1 correspondence via $h_i(c)$. Thus if (5.1) holds, we get that the number of orbits of $U_{\rho(J')}(\mathbb{F}_q) \times U_{\rho(I)}(\mathbb{F}_q)$ on

$$\mathcal{X}(I,J,\omega,\lambda,\mathbb{F}_q) \text{ is } (q-1)^{d+m} f^*(I,J,\omega,\lambda,q),$$

where $f^*(I,J,\omega,\lambda,q)$ is the number of orbits of

$$(U_{\rho(J')}(\mathbb{F}_q) \times U_{\rho(I)}(\mathbb{F}_q))_{\prod_{j=1}^{m} x_{a_j a_j}\lambda(1)}$$

on

$$\pi(I)EU_d(\mathbb{F}_q)\omega U_d(\mathbb{F}_q)\pi(J)^{-1} \cdot \prod_{j=1}^{m} x_{a_j,a_j}\lambda(1).$$

This result then leads on naturally to Theorem 1, so we first concentrate on proving that (5.1) holds.

The matrix M is sparse, and for such matrices, it is worthwhile to introduce a graph Γ. The vertex set of Γ is $[1,n] \times [1,d+m]$ and $(i,j),(k,l)$ are connected by an edge if and only if

$$\delta_{ik} + \delta_{jl} = 1 \text{ and } m_{ij}m_{kl} \neq 0.$$

The connected components of Γ are two types, types I and II. Type I components are defined to be those which contain a vertex (i,j) with $m_{ij} \neq 0$. Type II components are all the remaining components. Each of them consists of a single vertex (i,j) and $m_{ij} = 0$.

Consider a connected component $\tilde{\Gamma}$ of type I. Choose $(i,j) \in \tilde{\Gamma}$ with $m_{ij} \neq 0$, and suppose $(k,l) \in \tilde{\Gamma}$. There is a path between (i,j) and (k,l), and by consideration of the length of the path, we conclude that $m_{kl} \neq 0$ for all $(k,l) \in \tilde{\Gamma}$. In the case at hand, this means that $m_{kl} = 1$ or -1 for all $(k,l) \in \tilde{\Gamma}$.

Set

$$I(\tilde{\Gamma}) = \{k \in [1,n] | (k,l) \in \tilde{\Gamma} \text{ for some } l \in [1, d+m]\},$$

$$J(\tilde{\Gamma}) = \{l \in [1, d+m] | (k,l) \in \tilde{\Gamma} \text{ for some } k \in [1,n]\}.$$

Set

$$I(\tilde{\Gamma})' = [1,n] \backslash I(\tilde{\Gamma}), \quad J(\tilde{\Gamma})' = [1, d+m] \backslash J(\tilde{\Gamma}).$$

Then I argue that $m_{ij} = 0$ for all $(i,j) \in I(\tilde{\Gamma})' \times J(\tilde{\Gamma})$. Suppose $m_{ij} \neq 0$. By definition of $J(\tilde{\Gamma})$, there is $k \in [1,n]$ such that $(k,j) \in \tilde{\Gamma}$. For this element of $\tilde{\Gamma}$, we have $m_{kj} \neq 0$. But then (k,j) and (i,j) are connected in Γ, so $(i,j) \in \tilde{\Gamma}$, against $(i,j) \in I(\tilde{\Gamma})' \times J(\tilde{\Gamma})$. So $m_{ij} = 0$ if $(i,j) \in I(\tilde{\Gamma})' \times J(\tilde{\Gamma})$, and similarly, $m_{ij} = 0$ if $(i,j) \in I(\tilde{\Gamma}) \times J(\tilde{\Gamma})'$.

Set $r(\tilde{\Gamma}) = |I(\tilde{\Gamma})|, c(\tilde{\Gamma}) = |J(\tilde{\Gamma})|$. I proceed to show that $c(\tilde{\Gamma}) \leq r(\tilde{\Gamma})$. To do this, I partition $J(\tilde{\Gamma})$ into $J_1(\tilde{\Gamma}) = J(\tilde{\Gamma}) \cap [1,d]$, and $J_2(\tilde{\Gamma}) = J(\tilde{\Gamma}) \cap [d+1, d+m]$. I define a map

$$\tau : J(\tilde{\Gamma}) \to I(\tilde{\Gamma}),$$

as follows: if $j \in J_1(\tilde{\Gamma}), j\tau = j\pi_1^{-1} \in I \subseteq [1,n]$. Then $j\tau \in I(\tilde{\Gamma})$, since $(j\tau, j) \in \tilde{\Gamma}$, i.e., $m_{j\tau,j} \neq 0$. More precisely, $m_{j\tau,j} = -1$, since $j = (j\tau)\pi_1$. If $j \in J_2(\tilde{\Gamma})$, say $j = d+l$, set $j\tau = a_l \in I' \cap J$. Then $j\tau \in I(\tilde{\Gamma})$, since $(j\tau, j) \in \tilde{\Gamma}$. More precisely $m_{j\tau,j} = -1$, by definition of $m_{j\tau,j}$. Now this gives us our map τ. The restriction of τ to $J_1(\tilde{\Gamma})$ is an injection of $J_1(\tilde{\Gamma})$ into $I(\tilde{\Gamma}) \cap I$, since $\pi_1 \in S_n$. The restriction of τ to $J_2(\tilde{\Gamma})$ is an injection of $J_2(\tilde{\Gamma})$ into $I' \cap J$, since the map from $[1,m]$ to $I' \cap J$

given by $l \mapsto a_l$ is an injection. Since $I \cap (I' \cap J) = \phi, \tau$ is indeed an injection of $J(\tilde{\Gamma})$ into $I(\tilde{\Gamma})$, and so $c(\tilde{\Gamma}) \leq r(\tilde{\Gamma})$.

The preceding discussion shows that $[1, n]$ is partitioned into subsets $A_1, A_2, \ldots, A_r$, and $[1, d+m]$ is partitioned into subsets $B_1, \ldots, B_r$, with the following properties: Γ has precisely $r-1$ connected components of type I. They are $\Gamma_1, \ldots, \Gamma_{r-1}$, and

(i) $A_p = I(\Gamma_p), B_p = J(\Gamma_p), \quad 1 \leq p \leq r-1$;

(ii) If $i \in A_r$, then $m_{ij} = 0$ for all $j \in [1, d+m]$.

(iii) If $j \in B_r$, then $m_{ij} = 0$ for all $i \in [1, n]$.

We admit the possibility that $A_r = \phi$ and we admit the possibility that $B_r = \phi$.

I first show that $B_r = \phi$. For if $j \in [1, d]$ then $m_{j\pi_1^{-1}, j} \neq 0$, and if $j = d + \ell \in [d+1, d+m]$, then $m_{a_l, j} \neq 0$. Thus, to complete the proof of (5.1), it is necessary and sufficient to show that for all $p \in [1, r-1]$, the $\mathbb{Q}-$rank of M_p is $c(J_p)$ and all $\mathbb{Z}-$elementary divisors of M_p are 1, where M_p is the submatrix of M indexed by $I(\Gamma_p) \times J(\Gamma_p)$. Denote by $m_i(p)$ the i^{th} row of $M_p, i \in I(\Gamma_p)$.

I make use of (4.1), and show that

$$(5.2) \qquad I(\Gamma_p) \cap (C \cup D) \neq \phi, \quad \forall p \in [1, r-1].$$

Suppose false.

Let $J(\Gamma_p) \cap [d+1, d+m] = \{d + l_1, \ldots, d + l_s\}, l_1 < \cdots < l_s$. It may happen that $s = 0$, but I carry out discussion of all cases. Then $\{a_{l_1}, a_{l_2} \ldots, a_{l_s}\} \cup \{a_{l_1}\lambda, \ldots, a_{l_s}\lambda\} \subseteq I(\Gamma_p)$. Moreover, if $l \in [1, m]$, and $\{a_l, a_l\lambda\} \cap I(\Gamma_p) \neq \phi$, then $l \in \{l_1, \ldots, l_s\}$. Thus

$$A \cap I(\Gamma_p) = \{a_{l_1}, \ldots a_{l_s}\},$$

$$B \cap I(\Gamma_p) = \{a_{l_1}\lambda, \dots a_{l_s}\lambda\}.$$

By assumption, $C \cap I(\Gamma_p) = \phi, D \cap I(\Gamma_p) = \phi$, and so

$$(I' \cap J) \cap I(\Gamma_p) = \{a_{l_1}, \dots, a_{l_s}\},$$

$$(I \cap J') \cap I(\Gamma_p) = \{a_{l_1}\lambda, \dots, a_{l_s}\lambda\}.$$

Since $m_{ij} = 0$ for all $(i,j) \in I' \cap J' \times [1, d+m]$, we have

$$(I' \cap J') \cap I(\Gamma_p) = \phi.$$

Thus,

$$I(\Gamma_p) = \{a_{l_1}, \dots, a_{l_s}\} \,\dot{\cup}\, \{a_{l_1}\lambda, \dots, a_{l_s}\lambda\} \,\dot{\cup}\, I(\Gamma_p) \cap (I \cap J).$$

Denote by c the largest element of $I(\Gamma_p); c$ exists because $I(\Gamma_p)$ is a nonempty set of positive integers. Suppose $c \in \{a_{l_1} \dots, a_{l_s}\} \cup \{a_{l_1}\lambda, \dots, a_{l_s}\lambda\}$. Since λ is increasing, we get $c = a_{l_\mu}\lambda$ for some μ. So $c \in I \cap J'$, and $m_{c,c\pi_1} \neq 0$, whence $c\pi_1 \in [1,d] \cap J(\Gamma_p)$. Set $e = c\pi_1$. Then $m_{e\omega\pi_{2,}^{-1}e} \neq 0$, so $e\omega\pi_2^{-1} \in I(\Gamma_p)$. Since $e = c\pi_1$, we get $c\pi_1\omega\pi_2^{-1} \in I(\Gamma_p)$. But $c < c\pi_1\omega\pi_2^{-1}$, against the maximality of c in $I(\Gamma_p)$. So $c \in I(\Gamma_p) \cap I \cap J$. This also leads to a contradiction: since $c \in I$, we get $c\pi_1 \in J(\Gamma_p)$, and since $c\pi_1 \in J(\Gamma_p) \cap [1,d]$ we get $c\pi_1\omega\pi_2^{-1} \in I(\Gamma_p)$, and so $c < c\pi_1\omega\pi_2^{-1}$. This establishes (5.2).

Now suppose $I(\Gamma_p) \subseteq C \cup D$. In this case, each row of M_p has precisely one nonzero entry, which is 1 or -1, so every edge of Γ_p is a vertical segment. This implies that M_p is a 2×1 matrix of $\mathbb{Q}-$rank 1, whose unique $\mathbb{Z}-$elementary divisor is 1. Suppose $I(\Gamma_p) \nsubseteq C \cup D$. Set

$$\tilde{I}(\Gamma_p) = I(\Gamma_p) \backslash I(\Gamma_p) \cap (C \cup D),$$

and consider the submatrix $\tilde{M}(\Gamma_p)$ of M_p indexed by $\tilde{I}(\Gamma_p) \times J(\Gamma_p)$.

Now we view this setup in terms of integral lattices, by putting on $\mathbb{Z}^{d+m}$ the usual inner product: $(z, z') = \sum z_i z_i'$, if $z = (z_1, \ldots, z_{d+m}), z' \in (z_1', \ldots, z_{d+m}')$. Let $L(\Gamma_p)$ be the lattice generated by the rows of $\tilde{M}(\Gamma_p)$. By Witt's theorem, $L(\Gamma_p)$ is the orthogonal sum of sublattices $L_1(\Gamma_p), \ldots, L_k(\Gamma_p)$, each of which is of type A, D, or E. Since the E_6, E_7, E_8 lattices cannot be embedded isometrically in Z^N for any N, each $L_j(\Gamma_p)$ is of type A or D.

For each $j = 1, \ldots, k$, let $J(j) = \{i \in [1, d+m], e_i$ is not orthogonal to $L_j(\Gamma_p)\}$. First, suppose that the sets $J(1), \ldots, J(k)$ are pairwise disjoint. In this case, we get a partition of $I(\Gamma_p) \cap (C \cup D)$. For $i \in I(\Gamma_p) \cap (C \cup D)$, there is a unique $j \in [1, k]$ such that $m_i(p)$ is not orthogonal to $L_j(\Gamma_p)$. This forces $k = 1$, and it is trivial to check that $L_1(\Gamma_p) + \mathbb{Z} m_i(p) = \mathbb{Z}^{J(\Gamma_p)}$ where $i \in I(\Gamma_p) \cap (C \cup D)$. Thus, in this case, M_p has $\mathbb{Q}-$rank $|J(\Gamma_p)| = c(\Gamma_p)$, and all $\mathbb{Z}-$elementary divisors are 1.

It remains to treat the case where $k \geq 2$, and where $J(1) \cap J(2) \neq \phi$. To exclude this possibility, it is necessary to make use of the fact that for every $i \in \tilde{I}(\Gamma_p)$, the two nonzero entries of $m_i(p)$ are of opposite sign. So if $m_i(p) \in L_1(\Gamma_p), m_j(p) \in L_2(\Gamma_p)$, then on the one hand, $(m_i(p), m_j(p)) = 0$, whereas for suitable i, j, there is $l \in J(1) \cap J(2)$ with $(m_i(p), e_l) \neq 0, (m_j(p), e_l) \neq 0$. There are no solutions. It was necessary to discuss this case, since $D_2 \cong A_1 \oplus A_1$.

The isomorphism $D_3 \cong A_3$, causes no difficulty, since if $L_j(\Gamma_p) \cong D_3$, it is to be understood that $|J(j)| = 3$, and if $L_j(\Gamma_p) \cong A_3$, it is to be understood that $|J(j)| = 4$. So (5.1) holds.

6. $\Gamma^*(I, J, \lambda, F) = \Gamma(I, J, \lambda, F)$.

I retain the earlier notation:

$$I, J \in P_d([1,n]), \quad \omega \in C_n(I,J), \quad \lambda \in \Lambda(I,J),$$

$$(6.1) \qquad A = \mathcal{D}(\lambda), B = \mathcal{D}(\lambda)\lambda, C = I' \cap J \backslash A, D = I \cap J' \backslash B,$$

$$\pi_1 = \pi(I), \pi_2 = \pi(J), m = |\mathcal{D}(\lambda)|,$$

$$T = \prod_{a \in A} x_{a,a\lambda}(1).$$

Set

$$\mathcal{X} = \pi(I) E U_d(F) \omega U_d(F) \pi(J)^{-1} T.$$

Note that $\mathcal{X} = \mathcal{X}(I, J, \omega, \lambda, \xi_1, \ldots, \xi_d, t_{a_1}, \ldots t_{a_m}, F)$ is the special case $\xi_1 = \cdots = \xi_d = t_{a_1} = \cdots = t_{a_m} = 1$.

If $(v, u) \in (U_{\rho(J')}(F) \times U_{\rho(I)}(F))_T$, then

$$(6.2) \qquad v^{-1} T u = T = v T u^{-1}.$$

Pick $X \in \mathcal{X}$, so that

$$(6.3) \qquad X = \pi(I) E u_1 \omega u_2 \pi(J)^{-1} T, u_1, u_2 \in U_d(F).$$

By (6.2), we have

$$X = \pi(I) E u_1 \omega u_2 \pi(J)^{-1} v T u^{-1},$$

so

$$(6.4) \qquad X \circ \delta(u) = u^{-1} \pi(I) u_1 \omega u_2 \pi(J)^{-1} v T.$$

We have

$$u = x_2 x_1, x_2 \in U(R^+([1,n] \times I'), F), x_1 \in U(R^+(I \times I), F),$$

$$v = y_2 y_1, y_2 \in U(R^+(J' \times [1,n]), F), y_1 \in U(R^+(J \times J), F).$$

We have ordered $\sum^+$ by setting $r < s$ to mean $0 < s - r$. This is a linear ordering of $\sum^+$ and so there is a bijection

$$\rho_0 : [1,N] \to \sum{}^+ \tag{6.6}$$

such that $\rho_0(1) < \rho_0(2) < \cdots < \rho_0(N)$, where $N = |\sum^+|$. Since $\pi(I)$ agrees with $\lambda(I,[1,d])$ on I, it follows that if $r, s \in R^+(I \times I)$ and $r < s$, then $r\pi_1 < s\pi_1$, and, of course, $r\pi_1, s\pi_1 \in R^+([1,d] \times [1,d])$. A similar remark applies to $\pi(J)$ and $R^+(J \times J)$.

Write

$$\begin{aligned} x_1 &= \prod x_{\alpha,\beta}(t_{\alpha,\beta}), \\ y_1 &= \prod x_{\gamma,\delta}(t'_{\gamma,\delta}), \end{aligned} \tag{6.7}$$

where the product for x_1 runs over $R^+(I \times I)$ in ascending order, and the product for y_1 runs over $R^+(J \times J)$ in ascending order.

From (6.4), (6.5), (6.7), we get

$$\begin{aligned} X \circ \delta(u) &= x_1^{-1} x_2^{-1} \pi(I) E u_1 \omega u_2 \pi(J)^{-1} y_2 y_1 T \\ &= x_1^{-1} \pi(I) E u_1 \omega u_2 \pi(J)^{-1} y_1 T \\ &= \pi(I) E \{\prod x_{\alpha\pi_1,\beta\pi_1}(t_{\alpha,\beta})\}^{-1} u_1 \omega u_2. \\ &(\prod x_{\gamma\pi_2,\delta\pi_2}(t'_{\gamma\delta}))\pi(J)^{-1} T \in \mathcal{X}. \end{aligned} \tag{6.8}$$

This gives us a map

$$\begin{aligned} &\mathcal{X} \times (U_{\rho(J')}(F) \times U_{\rho(I)}(F))_T \to \mathcal{X}, \\ &(X, (v,u)) \mapsto X \circ \delta(u). \end{aligned} \tag{6.9}$$

As we have just seen, this map exists, and is given by (6.8). Since

$$(v_1, u_1) \cdot (v_2, u_2) = (v_1 v_2, u_1 u_2),$$

$$\delta(u_1 u_2) = \delta(u_1)\delta(u_2)$$

for all $(v_1, u_1), (v_2, u_2) \in (U_{\rho(J')}(F) \times U_{\rho(I)}(F))_T$, (6.9) gives us an action of $(U_{\rho(J')})F) \times U_{\rho(I)}(F))_T$ on $\mathcal{X}$.

Denote by $\Gamma^*(I, J, \lambda, F)$ the subgroup of $U_d(F) \times U_d(F)$ generated by all the elements

$$\left(\prod x_{\alpha\pi_1,\beta\pi_1}(t_{\alpha,\beta}), \prod x_{\gamma\pi_2,\delta\pi_2}(t'_{\gamma,\delta})\right) \tag{6.10}$$

such that

$$(y_2 y_1, x_2 x_1) \in (U_{\rho(J')}(F) \times U_{\rho(I)}(F))_T, \tag{6.11}$$

and

$$\begin{aligned} y_2 \in & U(R^+(J' \times [1,n]), F), x_2 \in (R^+([1,n] \times I'), F), \\ & x_1 = \prod x_{\alpha,\beta}(t_{\alpha,\beta}) \in U(R^+(I \times I), F), \\ & y_1 = \prod x_{\gamma,\delta}(t'_{\gamma,\delta}) \in U(R^+(J \times J), F). \end{aligned} \tag{6.12}$$

The action (6.9) shows us that the set of orbits of $(U_{\rho(J')}(F) \times U_{\rho(I)}(F))_T$ on $\mathcal{X}$ is in 1-1 correspondence with the set of orbits of $\Gamma^*(I, J, \lambda, F)$ on $U_d(F)\omega U_d(F)$, and so Theorem 1 is a consequence of

$$\Gamma^*(I, J, \lambda, F) = \Gamma(I, J, \lambda, F),$$

an equality which will be proved in this section.

Before attacking this problem directly, I make some comments about orderings. Let Ord be the set of all bijections $\rho : [1, N] \to \sum^+$, and let Ord^* consist of those ρ such that

$$r, s, r+s \in \sum\nolimits^+ \Rightarrow r = \rho(i), r+s = \rho(j) \text{ and } i < j.$$

Our given ρ_0 is in Ord^*. For obvious reasons, elements of Ord are called orderings of $\sum^+$.

If $u \in U(F)$ and $u \neq 1$, and

$$u = \prod_{i=1}^{N} x_{\rho_0(i)}(t_{\rho_0(i)}),$$

we define the leading ρ_0-root of u to be $\rho_0(i)$, where $t_{\rho_0(i)} \neq 0$ and $t_{\rho_0(j)} = 0$ for all $j < i$. We denote the leading ρ_0-root of u by $r_{\rho_0}(u)$, and set $r_{\rho_0}(1) = \infty$, with the convention that $r < \infty$ for all $r \in \sum^+$.

Lemma 6.1. *If $\rho \in Ord$, then to every $u \in U(F)$ is associated a unique map $\sum^+ \to F, r \mapsto t_r$ such that*

$$u = x_{\rho(1)}(t_{\rho(1)}) \cdot \ldots \cdot x_{\rho(N)}(t_{\rho(N)}).$$

Proof. Let S_u be the set of all sequences

$$\sigma = (x_{\rho(1)}(t_{\rho(1)}), y_1, \ldots, x_{\rho(N)}(t_{\rho(N)}), y_N),$$

where

(i) $t_{\rho(i)} \in F,$ all $i.$

(ii) $y_i \in U(F),$ all $i.$

(iii) $u = x_{\rho(1)}(t_{\rho(1)})y_1 \cdot \ldots \cdot x_{\rho(N)}(t_{\rho(N)})y_N.$

Then, $S_u \neq \phi$, since, for example, $(1, u, 1, \ldots, 1) \in S_u$. There are various maps $S_u \to S_u$ and various auxiliary sequences and integers associated to elements of S_u. In particular, we set $\sigma(\rho_0) = (r_{\rho_0}(y_1), \ldots, r_{\rho_0}(y_N))$, and we set $r(\sigma) = \min\{r_{\rho_0}(y_1), \ldots, r_{\rho_0}(y_N)\}$, where min is computed in the ρ_0-ordering.

We first concentrate on showing that $r(\sigma) = \infty$ for some $\sigma \in S_u$. Suppose false. Choose σ such that $r(\sigma)$ is maximal, and with this restriction, minimize the number $\nu(\sigma)$ of $i \in [1, N]$ such that $r_{\rho_0}(y_i) = r(\sigma)$.

Pick i such that $r_{\rho_0}(y_i) = r(\sigma)$, and set $r = r(\sigma)$. Write $y_i = x_r(t)\tilde{y}_i$, with $r < r_{\rho_0}(\tilde{y}_i)$. If $\rho(i) = r$, set $\tilde{\sigma} = (x_{\rho(1)}(t_{\rho_{(1)}}), y_1, \ldots, x_{\rho(i-1)}(t_{\rho(i-1)}), y_{i-1},$

$$x_{\rho(i)}(t_{\rho(i)} + t), \tilde{y}_i, x_{\rho(i+1)}(t_{\rho(i+1)}), \ldots),$$

and observe that $\tilde{\sigma} \in S_u$, and that either $r(\tilde{\sigma}) > r$, or $r(\tilde{\sigma}) = r$ and $\nu(\tilde{\sigma}) < \nu(\sigma)$, against our choice of σ. So we only need to rearrange σ, preserving $r(\sigma)$ and $\nu(\sigma)$ and reach the previous situation, to show that there is σ in S_u with $\rho(\sigma) = \infty$. This is easy to do, and the details are omitted.

If F is finite there is just one map for each u, since $|U_n(F)| = |F|^N$, the total number of maps from $\sum^+$ to F. If F is infinite, we make use of an elementary result. Namely, if R is any finitely generated subring of $F, a \in R$ and $a \neq 0$, then there is a finite field F_0 and a ring homomorphism $\varphi : R \to F_0$ such that $\varphi(a) \neq 0$. This yields Lemma 6.1.

Since Lemma 6.1 holds, we can now define $r_\rho(u)$ for all $\rho \in Ord, u \in U(F), u \neq 1$. We write

$$u = \prod_{i=1}^{N} x_{\rho(i)}(t_{\rho(i)})$$

and set $r_\rho(u) = \rho(i)$ if $t_{\rho(i)} \neq 0$ and $t_{\rho(j)} = 0$ for all $j < i$. And we set $r_\rho(1) = \infty$.

I also introduce the notation $r \underset{\rho}{<} s$ to mean that $r = \rho(i), s = \rho(j)$ and $i < j$. This agrees with the definition of $<$ for ρ_0, that $r \underset{\rho_0}{<} s$ if and only if $r < s$.

We amplify Lemma 6.1. Suppose $\rho_1 \in Ord^*$. Let $\Phi(\rho_1)$ be the set of all maps

$$\varphi : \sum\nolimits^+ \times F \to U(F) \tag{6.13}$$

such that

$$\begin{aligned} &(i) \quad \varphi(r, 0) = 1 \text{ for all } r \in \sum\nolimits^+. \\ &(ii) \quad \varphi(r, t) = x_r(t) y(r, t), \text{ where} \\ &\qquad\; r \underset{\rho_1}{<} r_{\rho_1}(y(r, t)), \text{ all } (r, t). \end{aligned}$$

Lemma 6.2. *Suppose $\rho \in Ord, \rho_1 \in Ord^*, u \in U(F)$. Then there is precisely one map $\sum^+ \to F, r \mapsto t_r$ such that $u = \varphi(\rho(1), t_{\rho(1)}) \ldots \varphi(\rho(N), t_{\rho(N)})$.*

Proof. We proceed as in the proof of Lemma 6.1. We again examine sequences

$$\sigma = (\varphi(\rho(1), t_{\rho(1)}), z_1, \ldots, \varphi(\rho(N), t_{\rho(N)}), z_N),$$

where $z_i \in U(F)$, and the product $\varphi(\rho(1), t_{\rho(1)}) z_1 \cdots = u$. This time, when we examine

$$\begin{aligned} p =& \varphi(\rho(i), t_{\rho(i)}) z_i \quad (z_i = x_r(t) \tilde{z}_i, \text{ etc.}) \\ =& \, x_{\rho(i)}(t_{\rho(i)}) y(\rho(i), t_{\rho(i)}) z_i, \end{aligned}$$

where $\rho(i) = r_{\rho_1}(z_1) = r(\sigma)$, we cannot simply amalgamate as before, but rather, write

$$\begin{aligned} p = & \, x_{\rho(i)}(t_{\rho(i)} + t) y(\rho(i), t_{\rho(i)} + t). \\ & \{y(\rho(i), t_{\rho(i)} + t))^{-1} x_r(t)^{-1} y(\rho(i), t_{\rho(i)}) z_i\}, \end{aligned}$$

and check that

$$\begin{aligned} r = & r(\sigma) = \rho(i) < r_{\rho_1}(y(\rho(i), t_{\rho(i)} + t)), \\ & r < r_{\rho_1}(x_r(t)^{-1} y(\rho(i), t_{\rho(i)}) x_r(t)), \\ & r < r_{\rho_1}(\tilde{z}_i). \end{aligned}$$

The proof then carries on as in Lemma 6.1. At the end of the proof a finitely generated subring of F appears, since we only need to augment the previous R by tossing into R all the elements of F which appear in expressions

$$y(\rho(i)), t_{\rho(i)}) = \prod_{j=i+1}^{N} x_{\rho(j)}(u_{i,\rho(j)}).$$

So Lemma 6.2 holds.

There is yet another game to play. Define a graph Γ whose vertex set is Ord, and where ρ_1, ρ_2 are connected by and edge if and only of there is $i \in [1, N-1]$ such that

$$\begin{aligned} (i) \quad & \rho_1(j) = \rho_2(j) \text{ for all } j \notin [1, N], j \in \{i, i+1\}. \\ (ii) \quad & \rho_1(i) = \rho_2(i+1), \rho_1(i+1) = \rho_2(i). \\ (iii) \quad & \rho_1(i) + \rho_1(i+1) \notin \sum\nolimits^{+}. \end{aligned}$$

I argue that for all maps $\sum^{+} \to F, r \mapsto t_r$,

$$\prod_{j=1}^{N} x_{\rho_1(j)}(t_{\rho_1(j)}) = \prod_{j=1}^{N} x_{\rho_2(j)}(t_{\rho_2(j)}). \tag{6.14}$$

Namely, the two products agree term by term except for the i^{th} and $(i+1)^{st}$ terms, which are

$$x_{\rho_1(i)}(t_{\rho_1(i)}) x_{\rho_1(i+1)}(t_{\rho_1(i+1)}), x_{\rho_2(i)}(t_{\rho_2(i)}) x_{\rho_2(i+1)}(t_{\rho_2(i+1)}),$$

respectively. Since

$$\begin{aligned} & x_{\rho_1(i)}(t_{\rho_1(i)}) x_{\rho_1(i+1)}(t_{\rho_1(i+1)}) = \\ & x_{\rho_2(i+1)}(t_{\rho_2(i+1)}) x_{\rho_2(i)}(t_{\rho_2(i)}), \end{aligned}$$

and since $\rho_1(i) + \rho_1(i+1) \notin \sum^{+}$, (6.14) follows.

Lemma 6.3. *Suppose $\rho \in Ord, \rho_1 \in Ord^*, R_1, R_2$ are non empty sets of positive roots and $\varphi : R_1 \to R_2$ has the following properties:*

(i) $r + \varphi(r) \in \sum^+$, *all* $r \in R_1$ *and* $r + s \notin \sum^+$ *if* $s \neq \varphi(r), r \in R_1, s \in R_2$.

(ii) *For all* $r_1 \in \sum^+$,

$$|\{(r,s) \in R_1 \times R_2 | r + s = r_1\}| \leq 1.$$

(iii) $r^* \underset{\rho_1}{\leq} r + \varphi(r)$ *for all* $r \in R_1$.

(iv) $r^* = r_0 + \varphi(r_o)$ *for some* $r_0 \in R_1$.

Suppose also that

$$x = \prod_{r \in R_1} x_r(t_r), y = \prod_{r \in R_2} x_r(u_r),$$

where both products are taken in ascending $\rho-$order, and where $t_r \neq 0$ for all $r \in R_1, u_r \neq 0$ for all $r \in R_2$. Then $r^* = r_{\rho_1}([x,y])$.

Proof. Define $R_{\alpha,\beta}$ for $\alpha, \beta \in \mathbb{N}$ by

$$R_{1,1} = \{r + s \in \sum^+ | r \in R_1, s \in R_2\},$$

$$R_{\alpha,\beta+1} = \{r + s \in \sum^+ | r \in R_{\alpha,\beta}, \quad s \in R_2\},$$

$$R_{\alpha+1,\beta} = \{r + s \in \sum^+ | r \in R_1, s \in R_{\alpha,\beta}\}.$$

Since $R_{1,1} = \{r + \varphi(r) | r \in R_1\}$, we get

$$r^* \underset{\rho_1}{\leq} r \text{ for all } r \in R_{1,1},$$

and since $\rho_1 \in Ord^*$, we get

$$r^* \underset{\rho_1}{<} r \text{ for all } r \in R_{\alpha,\beta} \text{ and all } \alpha, \beta \text{ with } \alpha + \beta > 2.$$

Now

$$[x,y] = \prod_{(r,s)\in R_1\times R_2} [x_r(t_r), x_s(u_s)] \cdot Z,$$

where $r^* \underset{\rho_1}{<} r_{\rho_1}(Z)$. Also

$$\prod_{(r,s)\in R_1\times R_2} [x_r(t_r), x_s(u_s)] = \prod_{r\in R_1} x_{r+\varphi(r)}(\pm t_r u_{\varphi(r)}).$$

The lemma follows.

Remark. It is easy to appreciate that in order to apply Lemma 6.3, very good information about R_1, R_2 needs to be available. The hypotheses of Lemma 6.3 are stringent.

Define $\overline{\rho} : [1, N] \to \sum^+$ as follows: if $r = e_i - e_j, s = e_k - e_l \in \sum^+$, then $r \underset{\overline{\rho}}{<} s$ if and only if one of the following holds:

(i) $\quad j < l.$

(ii) $\quad j = l$ and $i > k$.

One checks easily that $\overline{\rho} \in Ord^*$.

I begin the study of $\Gamma^*(I, J, \lambda, F)$ by partitioning $[1, n]^2$ into 38 subsets $S(1), \ldots, S(38)$, as in the following list

i	$S(i)$	i	$S(i)$
1	$I' \cap J' \times D$	21	$D \times I \cap J$
2	$I' \cap J' \times I \cap J$	22	$D \times A$
3	$C \times I \cap J$	23	$D \times C$
4	$I' \cap J' \times B$	24	$I' \cap J' \times C$
5	$\{(a, \alpha) \in A \times I \cap J \mid a\lambda > \alpha\}$	25	$B \times I' \cap J'$
6	$\{(a, a') \in A \times A \mid a\lambda > a'\lambda\}$	26	$D \times B$
7	$B \times B$	27	$D \times D$
8	$C \times A$	28	$D \times I' \cap J'$
9	$B \times D$	29	$I' \cap J' \times I' \cap J'$
10	$B \times I \cap J$	30	$I \cap J \times B$
11	$\{(a, \alpha) \in A \times I \cap J \mid a\lambda < \alpha\}$	31	$I \cap J \times D$
12	$\{(a, a') \in A \times A \mid a\lambda < a'\lambda\} \cup \{(a, a) \mid a \in A\}$	32	$I \cap J \times I' \cap J'$
13	$I' \cap J' \times A$	33	$A \times B$
14	$I \cap J \times I \cap J$	34	$A \times D$
15	$I \cap J \times A$	35	$A \times I' \cap J'$
16	$I \cap J \times C$	36	$C \times B$
17	$A \times C$	37	$C \times D$
18	$B \times A$	38	$C \times I' \cap J'$
19	$B \times C$		
20	$C \times C$		

To verify that we have partitioned $[1,n]^2$, note that $[1,n] = I \cap J \,\dot\cup\, A \,\dot\cup\, B \,\dot\cup\, C \,\dot\cup\, D \,\dot\cup\, I' \cap J'$ that $S(5) \cup S(11) = A \times (I \cap J), S(6) \cup S(12) = A \times A$, and that each of the remaining $X \times Y$, where $X, Y \in \{I \cap J, A, B, C, D, I' \cap J'\}$ occurs just once as an $S(i)$.

Next, I record that

$$(*) \qquad \begin{aligned} &e_a - e_{a'} \underset{\overline{\rho}}{\leq} e_{a\lambda} - e_{a'\lambda} \text{ if } a, a' \in A, a < a', a\lambda < a'\lambda \\ &e_\varepsilon - e_a \underset{\overline{\rho}}{\leq} e_\varepsilon - e_{a\lambda}, \text{ if } \varepsilon \in I \cap J, a \in A, \varepsilon < a \\ &e_{a\lambda} - e_{a'} \underset{\overline{\rho}}{\leq} e_{a\lambda} - e_{a'\lambda}, \text{ if } a, a' \in A, a\lambda < a'. \end{aligned}$$

I define $\varphi \in \Phi(\overline{\rho})$ as follows:

If $r \notin R^+(S(12)) \cup R^+(S(13)) \cup R^+(S(18))$,

$$\varphi(r,t) = x_r(t).$$

If $r = e_a - e_{a'} \in R^+(S(12))$,

$$\varphi(r,t) = x_r(t)x_{r'}(t), r' = e_{a\lambda} - e_{a'\lambda}.$$

If $r = e_\varepsilon - e_a \in R^+(S(13))$,

$$\varphi(r,t) = x_r(t)x_{r'}(-t), r' = e_\varepsilon - e_{a\lambda}.$$

If $r = e_{a\lambda} - e_{a'} \in R^+(S(18))$,

$$\varphi(r,t) = x_r(t)x_{r'}(-t), r' = e_{a\lambda} - e_{a'\lambda}.$$

From (*), we get $\varphi \in \Phi(\overline{\rho})$.

We construct $\tilde{\rho} \in Ord$. If $r \in R^+(S(i)), s \in R^+(S(j))$, we say $r \underset{\tilde{\rho}}{<} s$ if and only if one if the following holds:

$$(i) \quad i < j.$$

$$(ii) \quad i = j \text{ and } r \underset{\overline{\rho}}{<} s.$$

By Lemma 6.2, with $\overline{\rho}$ in the role of $\rho_1, \tilde{\rho}$ in the role of ρ, φ in the role of φ, we get that to each $u \in U(F)$ there is associated a unique map $\sum^+ \to F, r \mapsto t_r$, such that

$$u = \prod_{i=1}^{N} \varphi(\tilde{\rho}(i), t_{\tilde{\rho}(i)}). \tag{6.15}$$

Set

$$N_i = |R^+(S(i))|, \quad i \in [1, 38].$$

I examine the product (6.15) and in particular the contribution from the interval

$$I_{10,11} = \{N_1 + \cdots + N_9 + 1, N_1 + \cdots + N_{11}\}.$$

Set $R_{10,11} = R^+(S(10)) \cup R^+(S(11))$. We observe that

$$(R_{10,11} + R_{10,11}) \cap \sum^+ = \phi, \tag{6.16}$$

and so the order of the product

$$\prod_{i \in I_{10,11}} \varphi(\tilde{\rho}(i), t_{\tilde{\rho}(i)})$$

is immaterial. Here I am using (6.13) and also using $\varphi(\tilde{\rho}(i), t_{\tilde{\rho}(i)}) = x_{\tilde{\rho}(i)}(t_{\tilde{\rho}(i)})$ for all $i \in I_{10,11}$. If $i \in I_{10,11}$ and $i \leq N_1 + \cdots + N_{10}$, then $\tilde{\rho}(i) = e_{a\lambda} - e_\alpha$ for some

$a\lambda \in B, \alpha \in I \cap J, a\lambda < \alpha$. Then $e_a - e_\alpha \in R^+(S(11))$ and we have $e_a - e_\alpha = \tilde{\rho}(j)$ for some $j \in I_{10,11}, N_1 + \cdots + N_{10} < j$. We also have

$$x_{\tilde{\rho}(i)}(t_{\tilde{\rho}(i)}) \cdot x_{\tilde{\rho}(j)}(t_{\tilde{\rho}(j)}) = \tag{6.17}$$

$$x_{\tilde{\rho}(i)}(t_{\tilde{\rho}(i)}) - t_{\tilde{\rho}(j)}) \cdot x_{\tilde{\rho}(i)}(t_{\tilde{\rho}(j)}) x_{\tilde{\rho}(j)}(t_{\tilde{\rho}(j)}).$$

This gives rise to a map

$$\hat{x} : \sum\nolimits^{+} \times F \to U(F),$$

defined as follows:

$$\hat{x}(r,t) = \varphi(r,t) \text{ if } r \notin R^+(S(11)),$$

$$\hat{x}(r,t) = x_r(t)x_{r'}(t) \text{ if } r \in R^+(S(11)),$$

$$r = e_a - e_\alpha, r' = e_{a\lambda} - e_\alpha.$$

By (6.17), we see that to each $u \in U(F)$ there is associated a unique map $\sum^{+} \to F, r \mapsto t_r$ such that

$$u = \prod_{i=1}^{N} \hat{x}(\tilde{\rho}(i), t_{\tilde{\rho}(i)}). \tag{6.18}$$

The reason we cannot use Lemma 6.2 directly to get $\hat{x}$ is that in the $\overline{\rho}-$ordering, which is important here, we have

$$e_{a\lambda} - e_\alpha \underset{\overline{\rho}}{\leq} e_a - e_\alpha.$$

Were it not for (6.16), we would have an obstacle to getting (6.18). As it is, we get (6.18) simply by using (6.15) and then use (6.17) in the abelian group $U(R^+(S(10)) \cup R^+(S(11)), F)$ for each relevant pair $(\tilde{\rho}(i), \tilde{\rho}(j))$. Now set

$$K = \sum_{i=1}^{5} N_i, L = \sum_{i=1}^{10} N_i, \tag{6.19}$$

$$|R_{\rho(J')}| = M = \sum_{i=1}^{29} N_i, \tag{6.20}$$

$$\prod\nolimits_0^0 = \Big\{ \prod_{i=1}^{K} \hat{x}(\tilde{\rho}(i), t_{\tilde{\rho}(i)}) \Big\}, \tag{6.21}$$

$$\prod\nolimits_0^1 = \Big\{ \prod_{i=K+1}^{L} \hat{x}(\tilde{\rho}(i), t_{\tilde{\rho}(i)}) \Big\}, \tag{6.22}$$

$$\prod\nolimits_0 = \Big\{ \prod_{i=1}^{L} \hat{x}(\tilde{\rho}(i), t_{\tilde{\rho}(i)}) \Big\}, \tag{6.23}$$

$$\prod\nolimits_1 = \Big\{ \prod_{i=L+1}^{M} \hat{x}(\tilde{\rho}(i), t_{\tilde{\rho}(i)}) \Big\}. \tag{6.24}$$

Here it is to be understood that in defining any one of the sets in (6.21)-(6.24), we range over all maps from $\sum^+$ to F. The point of this fussiness is that $\prod_0^0, \prod_0, \prod_1$ are subgroups of $U_{\rho(J')}(F)$, that $\prod_0 = \prod_0^0 \cdot \prod_0^1, \prod_0^0 \cap \prod_0^1 = \{1\}, U_{\rho(J')}(F) = \prod_0 \prod_1, \prod_0 \cap \prod_1 = \{1\}$. The verification of these assertions is time-consuming, but utterly straightforward. The data have been arranged with considerable care, and taken in the right spirit, the verifications are fun.

Now let

$$\Gamma_0 = \prod\nolimits_0^T, \Gamma_1 = \prod\nolimits_1^T,$$

so that

$$U_{\rho(J')}(F)^T = \Gamma_0 \cdot \Gamma_1, \Gamma_0 \cap \Gamma_1 = \{1\}.$$

The situation has been cooked up so that

$$\Gamma_1 \subseteq U_{\rho(I)}(F), \tag{6.25}$$

yet another verification which is left to the reader. The reason the verification is so easy is that the $\hat{x}(\tilde{\rho}(i), t_{\tilde{\rho}(i)})$ are very special elements, and that

$$\hat{x}^T = \hat{x}[\hat{x}, T],$$

and the commutators may be calculated easily since $\{e_a - e_{a\lambda} | \alpha \in A\}$ is a set of pairwise orthogonal roots.

By (6.25), we get

$$U_{\rho(J')}(F)^T \cap U_{\rho(I)}(F) = \Gamma_1 \cdot \Delta,$$

where

$$\Delta = \Gamma_0 \cap U_{\rho(I)}(F).$$

The final piece of the puzzle falls into place once we show that $\Delta = 1$, so suppose by way of contradiction that $\Delta \neq 1$. Choose $x \in \Delta, x \neq 1$. Then there is $y \in \prod_0$ such that $x = y^T$, and we have

$$x \neq 1, x \in U_{\rho(I)}(F), x = y^T, y \in \prod\nolimits_0 . \tag{6.26}$$

Write $y = y_1 \dots y_{10}$, where $y_i \in U(R^+(S(i)), F)$, and set

$$Y = y_1 y_2 y_3 y_4 y_5, Z = y_6 y_7 y_8 y_9 y_{10}.$$

Since $U(R^+(S(i)), F) \subseteq C_{U(F)}(T), 1 \leq i \leq 5$, we get

$$x = y^T = Y \cdot Z^T.$$

Since $U(R^+(S(i)), F) \subseteq U_{\rho(I)}(F), 6 \leq i \leq 10$, we get

$$xZ^{-1} \in U_{\rho(I)}(F),$$

and

$$xZ^{-1} = YZ^TZ^{-1} = Y[T, Z^{-1}],$$

which we write as $Y \cdot [Z^{-1}, T]^{-1}$. Since $\prod_0^0 = \prod_0^{0T} \subseteq U_{\rho(I)\omega_0}(F)$, we get

$$Z^{-1} \notin \prod\nolimits_0^0.$$

Write $Z^{-1} = Z_0Z_1, Z_0 \in \prod_0^0, Z_1 \in \prod_0^1$, so that $Z_1 \neq 1$ and

$$Z_1 = \prod_{i \in E} x_{\tilde{\rho}(i)}(c_{\tilde{\rho}(i)}),$$

where E is a non empty subset of $\{K+1, \ldots, L\}$, and $c_{\tilde{\rho}(i)} \neq 0$ for all $i \in E$. Note that

$$[Z^{-1}, T] = [Z_0Z_1, T] = [Z_1, T] \tag{6.27}$$

as $[Z_0, T] = 1$. Set

$$R_1 = \{\tilde{\rho}(i) | i \in E\},$$

$$R_2 = \{e_a - e_{a\lambda} | a \in A\}.$$

We check that for each $r \in R_1$, there is precisely one s in R_2 such that $r + s \in \sum^+$.

Define

$$\varphi : R_1 \to R_2 \text{ by } r + \varphi(r) \in \sum\nolimits^+.$$

So φ is well-defined. We next check that (R_1, R_2, φ) satisfies the hypotheses of Lemma 6.3, with $\overline{\rho}$ in the role of $\rho_1, \tilde{\rho}$ in the role of ρ. By (6.24) and Lemma 6.3, there are $\tilde{\rho}(i), i \in E$, and $e_{a_0 - e_{a_0}\lambda} = \varphi(\tilde{\rho}(i))$ such that

$$r_{\overline{\rho}}([Z^{-1}, T]^{-1}) = \tilde{\rho}(i) + \varphi(\tilde{\rho}(i)).$$

Yet another check reveals that

$$\tilde{\rho}(i) + \varphi(\tilde{\rho}(i)) \notin \bigcup_{i=1}^{5} R^{+}(S(i)),$$

and we conclude that

$$r_{\overline{\rho}}(Y[Z^{-1}, T]^{-1}) \in \{\tilde{\rho}(i) + \varphi(\tilde{(i)})\} \cup \bigcup_{i=1}^{5} R^{+}(S(i)).$$

This is false, since

$$\{\tilde{\rho}(i) + \varphi(\tilde{\rho}(i))\} \cup \bigcup_{i=1}^{5} R^{+}(S(i)) \subseteq R_{\rho(I)\omega_0}.$$

All the pieces fit snugly and we have shown that

$$U_{\rho(J')}(F)^{T} \cap U_{\rho(I)}(F) = \Gamma_1.$$

Since

$$(U_{\rho(J')}(F) \times U_{\rho(I)}(F)_T = \{(TuT^{-1}, u) | u \in \Gamma_1\},$$

the last check reveals that $\Gamma^{*}(I, J, \lambda, F) = \Gamma(I, J, \lambda, F)$, by appealing to (6.10), (6.11), (6.12), (1.33)-(1.38).

As a help to the reader, and as evidence of the ease with which we check that $\Gamma^{*}(I, J, \lambda, F) = \Gamma(I, J, \lambda, F)$, I provide the necessary checks.

We have

$$\prod\nolimits_1 = \{ \prod_{i=L+1}^{M} \hat{x}(\tilde{\rho}(i), t_{\tilde{\rho}(i)})\},$$

where we range over all maps from $\sum^{+}$ to F. Fix i with $L+1 \leq i \leq M$, and consider the map which sends $\tilde{\rho}(i)$ to t and sends $\tilde{\rho}(j)$ to 0 for all $j \neq i$. This shows that $\prod_1$ contains $\hat{x}(\tilde{\rho}(i), t)$ for all $i \in \{L+1, \ldots, M\}$ and all $t \in F$. We examine all these special elements.

Case 1. $\tilde{\rho}(i) = r$ and $L+1 \leq i \leq L+N_{11}$. Here we have $r = e_a - e_\alpha, a \in A, \alpha \in I \cap J, a < \alpha$, and $r' = e_{a\lambda} - e_\alpha, a\lambda < \alpha, \hat{x}(r,t) = x_r(t)x_{r'}(t)$. Here I have used the property that λ is increasing to conclude from $a\lambda < \alpha$ that $a < \alpha$. Thus, $(x_r(t)x_{r'}(t))^T = u \in \Gamma_1$. Since

$$T = x_{a,a\lambda}(1) \cdot T_1 = T_1 \cdot x_{a,a\lambda}(1),$$

where T_1 centralizes $x_r(t)x_{r'}(t)$, we get

$$\begin{aligned} u &= (x_r(t)x_{r'}(t))^{x_{a,a\lambda}(1)} \\ &= x_r(t) \cdot x_{r'}(t)^{x_{a,a\lambda}(1)} \\ &= x_r(t) \cdot x_{r'}(t)[x_{r'}(t), x_{a,a\lambda}(1)]. \end{aligned}$$

Now $x_{r'}(t) = x_{a\lambda.\alpha}(t)$, and so

$$u = x_r(t)x_{r'}(t) \cdot x_{a,\alpha}(-t) = x_{r'}(t).$$

Also $a\lambda \in I \cap J', \alpha \in I \cap J$, so $(a\lambda, \alpha) \in I \times I$. Thus, setting $v = x_r(t)x_{r'}(t) = TuT^{-1}$, we have $(v, u) \in (U_{\rho(J')}(F) \times U_{\rho(I)})_T$, and

$$u = x_2 x_1, v = y_2 y_1,$$

where $x_2 = 1, x_1 = x_{r'}(t)$. As for v, we have $a \in I' \cap J, \alpha \in I \cap J, a\lambda \in I \cap J'$, so

$$x_r(t) \in U(R^+(J \times J), F), \quad x_{r'}(t) \in U(R^+(J' \times J), F),$$

and $y_2 = x_{r'}(t), y_1 = x_r(t)$. So $x_1 = x_{a\lambda,\alpha}(t), y_1 = x_{a,\alpha}(t)$, and

$$(x_{a\lambda\pi_1,\alpha\pi_1}(t), x_{a\pi_2,\alpha\pi_2}(t)) \in \Gamma^*(I, J, \lambda, F).$$

From (1.37), we get

$$(x_{a\lambda\pi_1,\alpha\pi_1}(t), x_{a\pi_1,\alpha\pi_2}(t)) \in \Gamma(I, J, \lambda, F),$$

and this holds for all $\tilde{\rho}(i) \in e_a - e_\alpha$ with $L + 1 \leq i \leq L + N_{11}$, that is, for all $r \in R^+(S(11))$, and for all $t \in F$.

Case 2. $L + N_{11} + 1 \leq i \leq L + N_{11} + N_{12}$.

Here $\tilde{\rho}(i) = r, r = e_a - e_{a'}, a < a', a, a' \in A, a\lambda < a'\lambda$. Also

$$\begin{aligned}\hat{x}(\tilde{\rho}(i), t) &= \varphi(\tilde{\rho}(i), t) \\ &= x_r(t)x_{r'}(t), r' = e_{a\lambda} - e_{a'\lambda}.\end{aligned}$$

We need $u = (x_r(t)x_{r'}(t))^T$. Now $T = x_{a,a\lambda}(1)x_{a',a'\lambda}(1)T_1$, where T_1 centralizes $x_r(t)$ and $x_{r'}(t)$, so

$$\begin{aligned}u &= x_{a,a'}(t)^{x_{a,a\lambda}x_{a',a'\lambda}(1)} \cdot_{x_a\lambda,a'\lambda} (t)^{x_{a,a\lambda}(1)x_{a',a'\lambda}(1)} \\ &= x_{a,a'}(t)^{x_{a',a'\lambda}(1)} \cdot x_{a\lambda,a'\lambda}(t)^{x_{a,a\lambda}(1)} \\ &= x_{a,a'}(t)[x_{a,a'}(t), x_{a',a'\lambda}(1)] \cdot x_{a\lambda,a'\lambda}(t)[x_{a\lambda,a'\lambda}(t), x_{a,a\lambda}(1)] \\ &= x_{a,a'}(t)x_{a,a'\lambda}(t) \cdot x_{a\lambda,a'\lambda}(t)x_{a,a'\lambda}(-t) \\ &= x_{a,a'}(t)x_{a\lambda,a'\lambda}(t) = u.\end{aligned}$$

Thus,

$$(x_{a,a'}(t)x_{a\lambda,a'\lambda}(t), x_{a,a'}(t)x_{a\lambda,a'\lambda}(t)) \in (U_{\rho(J')}(F) \times U_{\rho(I)}(F))_T.$$

We have $a \in I' \cap J, a' \in I' \cap J$, so $x_{a,a'}(t) \in U(R^+(I' \times I'), F)$; and $a\lambda \in I \cap J', a'\lambda \in I \cap J'$, so $x_{a\lambda,a'\lambda}(t) \in U(R^+(I \times I), F)$. Hence $x_2 = x_{a,a'}(t), x_1 = x_{a\lambda,a'\lambda}(t)$.

Also,

$$x_{a,a'}(t) \in U(R^+(J \times J), F),$$

$$x_{a\lambda,a'\lambda}(t) \in U(R^+(J' \times J'), F),$$

so $y_2 = x_{a\lambda,a'\lambda}(t), y_1 = x_{a,a'}(t)$. Hence

$$x_1 = x_{a\lambda,a'\lambda}(t), y_1 = x_{a,a'}(t),$$

and so

$$(x_{a\lambda\pi_1,a'\lambda\pi_1}(t), x_{a\pi_2,a'\pi_2}(t)) \in \Gamma^*(I, J, \lambda, F).$$

From (1.36), we get

$$(x_{a\lambda\pi_1,a'\lambda\pi_1}(t), x_{a\pi_2,a'\pi_2}(t)) \in \Gamma(I, J, \lambda, F),$$

for all $t \in F$ and all $e_a - e_{a'} \in R^+(S(12))$.

Case 3. $\tilde{\rho}(i) = r$ and $L + N_{11} + N_{12} + 1 \leq i \leq L + N_{11} + N_{12} + N_{13}$.

Here $r = e_\epsilon - e_a$, where $\epsilon \in I' \cap J', \quad a \in A, \epsilon < a$, and

$$\hat{x}(r,t) = \varphi(r,t) = x_r(t)x_{r'}(-t), \quad r' = e_\epsilon - e_{a\lambda},$$

so

$$u = (x_r(t)x_{r'}(-t))^T.$$

We have $T = x_{a,a\lambda}(1) \cdot T_1 = T_1 \cdot x_{a,a\lambda}(1)$, where T_1 centralizes $x_r(t)$ and $x_{r'}(-t)$,

so

$$\begin{aligned} u &= (x_{\epsilon,a}(t)x_{\epsilon,a\lambda}(-t))^{x_{a,a\lambda}(1)} \\ &= x_{\epsilon,a}(t)[x_{\epsilon,a}(t), x_{a,a\lambda}(1)] \cdot x_{\epsilon,a\lambda}(-t) \\ &= x_{\epsilon,a}(t)x_{\epsilon,a\lambda}(t)x_{\epsilon,a\lambda}(-t) = x_{\epsilon,a}(t). \end{aligned}$$

Now $e_\epsilon - e_a \in R^+(I' \times I')$, and $e_\epsilon - a_a \in R^+(J' \times J), e_\epsilon - e_{a\lambda} \in R^+(J' \times J')$, so

$$x_2 = x_{\epsilon,a}(t), x_1 = 1, y_2 = x_{\epsilon,a}(t)x_{\epsilon,a\lambda}(-t), \quad y_1 = 1,$$

so, $x_1 = 1, y_1 = 1$, and the contribution to $\Gamma^*(I, J, \lambda, F)$ is (1,1).

Case 4. $\tilde{\rho}(i) = r$ and $r \in R^+(S(14))$.

Here $\hat{x}(r,t) = \varphi(r,t) = x_r(t)$, and $r = e_j - e_{j'}, j, j' \in I \cap J, j < j', u = x_r(t)^T = x_r(t)$, so $\upsilon = u = x_r(t)$, and $x_2 = 1, x_1 = x_r(t), y_2 = 1, y_1 = x_r(t)$, whence $(x_{j\pi_1, j'\pi_1}(t), x_{j'\pi_1}(t)) \in \Gamma^*(I, J, \lambda, F)$. By (1.33), we get

$$(x_{j\pi_1, j'\pi_1}, (t), x_{j\pi_2, j'\pi_2}(t)) \in \Gamma(I, J, \lambda, F).$$

Case 5. $\tilde{\rho}(i) = r$ and $r \in R^+(S(15))$.

Here $\hat{x}(r,t) = \varphi(r,t) = x_r(t), r = e_i - e_a, i \in I \cap J, a \in A, i < a$,

$$u = x_r(t)^T = x_{i,a}(t)^{x_{a,a\lambda}(1)} = x_{i,a}(t) x_{i,a\lambda}(t),$$

$e_i - e_a \in R^+(I \times I'), e_i - e_{a\lambda} \in R^+(I \times I)$, so $x_2 = x_{i,a}(t), x_1 = x_{i,a\lambda}(t); e_i - e_a \in R^+(J \times J)$, so $y_2 = 1, y_1 = x_{i,a}(t)$, and $x_1 = x_{i,a\lambda}(t), y_1 = x_{i,a}(t)$,

$$(x_{i\pi_1, a\lambda\pi_1}(t)), x_{i\pi_2, a\pi_2}(t)), \in \Gamma^*(I, J, \lambda, F).$$

By(1.38)

$$(x_{i\pi_1, a\lambda\pi_1}(t), x_{i\pi_2, a\pi_2}(t)) \in \Gamma(I, J, \lambda, F).$$

Case 6. $\tilde{\rho}(i) = r \in R^+(S(16)), r = e_j - e_\gamma, j \in I \cap J, \gamma \in C$. Also, $\hat{x}(r,t) = \varphi(r,t) = x_r(t), u = u_r(t)^T = x_r(t)$. Since $C \subseteq I' \cap J$, we get $u \in R^+(I \times I')$, so $x_2 = x_r(t), x_1 = 1$. Also, $x_r(t) \in R^+(J \times J), F$, so $y_2 = 1, y_1 = x_r(t) = x_{j,\gamma}(t)$, and

$$(1, x_{j\pi_2, \gamma\pi_2}(t)) \in \Gamma^*(I, J, \lambda, F).$$

By (1.34),

$$(1, x_{j\pi_2\gamma\pi_2}(t)) \in \Gamma(I, J, \lambda, F).$$

Case 7. $\tilde{\rho}(i) = r \in R^+(S(17)), r = e_a - e_\gamma, \quad a \in A, \gamma \in C, a < \gamma, \quad \hat{x}(r,t) = \varphi(r,t) = x_r(t), u = x_r(t)^T = x_r(t), x_r(t) \in R^+(I' = \times I')$, so $x_2 = x_r(t), x_1 = 1$. Since $x_r(t) \in R^+(J \times J)$, we have $y_2 = 1, y_1 = x_r(t)$, so

$$x_1 = 1, y_1 = x_{a,\gamma}(t),$$

and

$$(1, x_{a\pi_2,\gamma\pi_2}(t)) \in \Gamma^*(I, J, \lambda, F).$$

By (1.34), we have

$$(1, x_{a\pi_2,\gamma\pi_2}(t)) \in \Gamma(I, J, \lambda, F).$$

Case 8. $\tilde{\rho}(i) = r \in R^+(S(18)), r = e_{a\lambda} - e_{a'}, a, a' \in A, a\lambda < a'$, and $\hat{x}(r,t) = \varphi(r,t) = x_r(t)x_{r'}(-t), r' = e_{a\lambda} - e_{a'\lambda}$. Now $u = (x_r(t)x_{r'}(-t))^T, T = x_{a,a\lambda}(1)x_{a',a'\lambda}(1)T_1$, and T_1 centralizes $x_r(t)$ and $x_{r'}(-t)$, so

$$\begin{aligned}
u &= (x_r(t)x_{r'}(-t))^{x_{a,a\lambda}(1)x_{a',a'\lambda}(1)} \\
&= x_{a\lambda,a'}(t)^{x_{a,a\lambda}(1)x_{a',a'\lambda}(1)} \cdot x_{a\lambda,a'\lambda}(-t)^{x_{a,a\lambda}(1)} \\
&= (x_{a\lambda,a'}(t)[x_{a\lambda,a'}(t), x_{a,a\lambda}(1)])^{x_{a',a'\lambda}(1)} \cdot x_{a\lambda,a'\lambda}(-t)[x_{a\lambda,a'\lambda}(-t), x_{a,a\lambda}(1)] \\
&= (x_{a\lambda,a'}(t)x_{a,a'}(t)(-t))^{x_{a,a'\lambda}(1)} \cdot x_{a\lambda,a'\lambda}(-t)x_{a,a'\lambda}(t) \\
&= x_{a\lambda,a'}(t)[x_{a\lambda,a'}(t), x_{a',a'\lambda}(1)] \cdot x_{a,a'}(-t)[x_{a,a'}(-t), x_{a',a'\lambda}(1)] \cdot \\
&\quad x_{a\lambda,a'\lambda}(-t)x_{a,a'\lambda}(t) \\
&= x_{a\lambda,a'}(t)x_{a\lambda,a'\lambda}(t)x_{a,a'}(-t)x_{a',a'\lambda}(-t)x_{a\lambda,a'\lambda}(-t)x_{a,a'\lambda}(t) \\
&= x_{a\lambda,a'}(t)x_{a,a'}(-t).
\end{aligned}$$

Now $x_{a\lambda,a'}(t) \in U(R^+(I \times I'), F), x_{a,a'}(t) \in U(R^+(I' \times I'), F)$, so $u = x_2x_1, x_2 = x_{a\lambda,a'}(t)x_{a,a'}(-t), x_1 = 1, \quad TuT^{-1} = v = y_2y_1 = x_{a\lambda,a'}(t)x_{a\lambda,a'\lambda}(t)$ where $x_{a\lambda,a'}(t) \in$

$U(R^+(J'\times J),F), x_{a\lambda,a'\lambda}(-t)\in U(R^+(J'\times J'),F)$, so $y_2 = x_{a\lambda,a'}(t)x_{a\lambda,a'\lambda}(-t), y_1 =$ 1. Hence

$$x_1 = 1, y_1 = 1,$$

and the $\Gamma^*(I,J,\lambda,F)-$contribution is (1,1).

Case 9. $\tilde{\rho}(i) = r \in R^+(S(19)), r = e_{a\lambda} - e_\gamma, a \in A, \gamma \in C, a\lambda < \gamma, \hat{x}(r,t) = x_r(t), u = x_r(t)^T = x_{a\lambda,\gamma}(t)^{x_{a,a\lambda}(1)} = x_{a\lambda,\gamma}(t)[x_{a\lambda,\gamma}(t), x_{a,a\lambda}(1)] = x_{a\lambda,\gamma}(t)x_{a,\gamma}(-t),$ and

$$x_{a\lambda,\gamma}(t)\in U(R^+(I\times I'),F),$$

$$x_{a,\gamma}(-t)\in U(R^+(I'\times I'),F),$$

so $u = x_2x_1, x_2 = x_{a\lambda,\gamma}(t)x_{a,\gamma}(-t), x_1 = 1$, and $\upsilon = TuT^{-1} = x_r(t), x_{a\lambda,\gamma}(t) \in U(R^+(J'\times J),F)$, so $\upsilon = y_2y_1, y_2 = x_{a\lambda,\gamma}(t), y_1 = 1$, and

$$x_1 = 1, y_1 = 1,$$

and the $\Gamma^*(I,J,\lambda,F)-$contribution is (1,1).

Case 10. $\tilde{\rho}(i) = r \in R^+(S(20)), r = e_\gamma - e_{\gamma'}, \gamma, \gamma' \in C, \gamma < \gamma', \hat{x}(r,t) = x_{\gamma,\gamma'}(t), u = x_{\gamma,\gamma,}(t)^T x_{\gamma,\gamma'}(t)$. Since $x_{\gamma,\gamma'}(t) \in U(R^+(I'\times I'),F)$, we have $u = x_2x_1, x_2 = x_{\gamma,\gamma'}(t), x_1 = 1$. Since $x_{\gamma,\gamma'}(t) \in U(R^+(J\times J),F)$, we have $\upsilon = y_2y_1, y_2 = 1, y_1 = x_{\gamma,\gamma'}(t)$. So $x_1 = 1, y_1 = x_{\gamma,\gamma'}(t)$ and

$$(1, x_{\gamma\pi_2,\gamma'\pi_2}(t)) \in \Gamma^*(I,J,\lambda,F).$$

By (1.34)

$$(1, x_{\gamma\pi_2,\gamma'\pi_2}(t)) \in \Gamma(I,J,\lambda,F).$$

Case 11. $\tilde{\rho}(i) = r \in R^+(S(21)), r = e_\delta - e_j, \quad \delta \in D, \quad j \in I \cap J, \quad \hat{x}(r,t) = x_r(t), u = x_r(t)^T = x_r(t)$. Since $x_{\delta,j}(t) \in U(R^+(I \times I), F)$, we have

$$u = x_2 x_1, x_2 = 1, x_1 = x_{\delta,j}(t).$$

Since $x_{\delta,j}(t) \in U(R^+(I \times I), F)$, we have

$$TuT^{-1} = \upsilon = y_2 y_1, y_2 = x_{\delta_j}(t), y_1 = 1,$$

$$x_1 = x_{\delta,j}(t), y_1 = 1,$$

and

$$(x_{\delta\pi_1, j\pi_1}(t), 1) \in \Gamma^*(I, j, \lambda, F).$$

By (1.31),

$$(x_{\delta\pi_1, j\pi_1}, (t), 1) \in \Gamma(I, J, \lambda, F).$$

Case 12. $\tilde{\rho}(i) = r \in R^+(S(22)), r = e_\delta - e_a, \quad \delta \in D, \quad a \in A, \quad \delta < a, \quad \hat{x}(r,t) = x_{\delta,a}(t), u = x_{\delta,a}(t)^T = x_{\delta,a}(t) x_{\delta,a\lambda}(t)$. Since $x_{\delta,a}(t) \in U(R^+(I \times I'), F), x_{\delta,a\lambda}(t) \in U(R^+(I \times I), F)$, we have $u = x_2 x_1, x_2 = x_{\delta,a}(t), x_1 = x_{\delta,a\lambda}(t)$. Since $x_{\delta,a}(t) \in U(R^+(J' \times J), F)$, we have

$$TuT^{-1} = \upsilon = x_{\delta,a}(t) = y_2 y_1, y_2 = x_{\delta,a}(t), y_1 = 1,$$

so

$$x_1 = x_{\delta,a\lambda}(t), y_1 = 1,$$

and

$$(x_{\delta\pi_1, a\lambda\pi_1}(t), 1) \in \Gamma^*(I, J, \lambda, F).$$

By (1.33),

$$(x_{\delta\pi_1, a\lambda\pi_1}(t), 1) \in \Gamma(I, J, \lambda.F).$$

Case 13. $\tilde{\rho}(i) = r \in R^+(S(23)), r = e_\delta - e_\gamma, \quad \delta \in D, \quad \gamma \in C, \quad \delta < \gamma, \quad \hat{x}(r,t) = x_{\delta,\gamma}(t), u = x_{\delta,\gamma}(t)^T = x_{\delta,\gamma}(t)$. Since $x_{\delta,\gamma}(t) \in U(R^+(I \times I'), F)$, we have

$$u = x_2 x_1, x_2 = x_{\delta,\gamma}(t), x_1 = 1.$$

Since $x_{\delta,\gamma}(t) \in U(R^+(J' \times J), F)$, we have

$$TyT^{-1} = \upsilon = y_2 y_1, y_2 = x_{\delta\gamma}(t), y_1 = 1,$$

so

$$x_1 = 1, y_1 = 1,$$

and the $\Gamma^*(I, J, \lambda, F)-$contribution is (1, 1).

Case 14. $\tilde{\rho}(i) = r \in R^+(S(24)), r = e_\varepsilon - e_\gamma, \quad \varepsilon \in I' \cap J', \quad \gamma \in C, \quad \varepsilon < \gamma, \quad \hat{x}(r,t) = x_{\varepsilon,\gamma}(t), u = x_{\varepsilon,\gamma}(t)^T = x_{\varepsilon,\gamma}(t)$. Since $x_{\varepsilon,\gamma}(t) \in U(R^+(I' \times I'), F)$, we have

$$u = x_2 x_1, x_2 = x_{\varepsilon,\gamma}(t), x_1 = 1.$$

Since $x_{\varepsilon,\gamma}(t) \in U(R^+(J' \times J), F)$, we have

$$x_{\varepsilon,\gamma}(t) = y_2 y_1, \quad y_2 = x_{\varepsilon,\gamma}(t), \quad y_1 = 1,$$

so $x_1 = 1, y_1 = 1$, and the $\Gamma^*(I, J, \lambda, F)-$contribution is (1, 1).

Case 15. $\tilde{\rho}(i) = r \in R^+(S(25)), r = e_{a\lambda} - e_\varepsilon, a \in A, \quad \varepsilon \in I' \cap J', \quad a\lambda < \varepsilon, \quad \hat{x}(r,t) = x_{a\lambda,\varepsilon}(t), u = x_{a\lambda,\varepsilon}(t)^T = x_{a\lambda,\varepsilon}(t)^{x_{a,a\lambda}(1)} = x_{a\lambda,\varepsilon}(t) x_{a,\varepsilon}(-t)$. Since $x_{a\lambda,\epsilon}(t) \in U(R^+(I \times I'), F), x_{a,\epsilon}(-t) \in U(R^+(I' \times I'), F)$, we have $u = x_2 x_1, x_2 = x_{a\lambda,\epsilon}(t) x_{a\epsilon}(-t), x_1 = 1$. Since $x_{a\lambda,\epsilon}(t) \in U(R^+(J' \times J'), F)$, we have $\upsilon = y_2 y_1, y_2 = x_{a\lambda,\epsilon}(t), y_1 = 1$, so $x_1 = 1, y_1 = 1$ and the $\Gamma^*(I, J, \lambda, F)-$contribution is (1, 1).

Case 16. $\tilde{\rho}(i) = r \in R^+(S(26)), r = e_\delta - e_{a\lambda}, \quad \delta \in D, \quad a \in A, \quad \delta < a\lambda, \quad \hat{x}(r,t) = x_{\delta,a\lambda}(t), u = x_{\delta,a\lambda}(t)^T = x_{\delta,a\lambda}(t)$. Since $x_{\delta,a\lambda}(t) \in U(R^+(I \times I), F)$, we have $u = x_2 x_1, x_2 = 1, x_1 = x_{\delta,a\lambda}(t)$; since $x_{\delta,a\lambda}(t) \in U(R^+(J \times J'),\ F)$, we have

$$TuT^{-1} = v = y_2 y_1, y_2 = x_{\delta,a\lambda}(t), y_1 = 1,$$

so $x_1 = x_{\delta,a\lambda}(t), \quad y_1 = 1,$

$$(x_{\delta\pi_1,a\lambda\pi_1}(t), 1) \in \Gamma^*(I, J, \lambda, F).$$

By (1.33),

$$(x_{\delta\pi_1,a\lambda\pi_1}(t), 1) \in \quad \Gamma(I, J, \lambda, F).$$

Case 17. $\tilde{\rho}(i) = r \in R^+(S(27)), r = e_\delta - e_{\delta'}, \delta, \delta' \in D, \quad \delta' < \delta, \hat{x}(r,t) = x_{\delta,\delta'}(t), u = x_{\delta,\delta'}(t)^T$. Since $x_{\delta,\delta'}(t) \in U(R^+(I \times I), F)$, we have $u = x_2 x_1, x_2 = 1, x_1 = x_{\delta,\delta'}(t)$; since $x_{\delta,\delta'}(t) \in U(R^+(J' \times J'), F)$, we have $TuT^{-1} = v = y_2 y_1, y_2 = x_{\delta,\delta'}(t), y_1 = 1$, so $x_1 = x_{\delta,\delta'}(t), y_1 = 1$, and

$$(x_{\delta\pi_1,\delta'\pi_1}(t), 1) \in \Gamma^*(I, J, \lambda, F).$$

By (1.33),

$$(x_{\delta\pi_1,\delta'\pi_1}(t), 1) \in \Gamma(I, J, \lambda, F).$$

Case 18. $\tilde{\rho}(i) = r \in R^+(S(28)), r = e_\delta - e_\varepsilon, \quad \delta \in D, \quad \varepsilon \in I' \cap J', \quad \delta < \varepsilon, \quad \hat{x}(r,t) = x_{\delta,\varepsilon}(t), \ u = x_{\delta,\varepsilon}(t)^T = x_{\delta,\varepsilon}(t)$. Since $x_{\delta,\varepsilon}(t) \in U(R^+(I \times I'), F)$, we have $TuT^{-1} = v = y_2 y_1, \quad y_2 = x_{\delta,\varepsilon}(t), y_1 = 1$, so $x_1 = 1, y_1 = 1$, and the $\Gamma^*(I, J, \lambda, F)-$contribution is (1,1).

Case 19. $\tilde{\rho}(i) = r \in R^+(S(29)), r = e_\varepsilon - e_{\varepsilon'} \quad \varepsilon, \varepsilon' \in I' \cap J', \quad \varepsilon < \varepsilon', \quad \hat{x}(r,t) = x_r(t), u = x_{\varepsilon,\varepsilon'}(t)^T = x_{\varepsilon,\varepsilon'}(t) \in U(R^+(I' \times I', F)), u = x_2x_1, x_2 = x_{\epsilon,\varepsilon'}(t), x_1 = 1, \quad TuT^{-1} = v = y_2y_1, \quad y_2 = x_{\varepsilon,\varepsilon'}(t), \quad y_1 = 1,$ so the $\Gamma^*(I, J, \lambda, F)-$contribution is (1, 1).

Thus, in each case, the $\Gamma^*(I, J, \lambda, F)-$contribution is contained in $\Gamma(I, J, \lambda, F)$. Conversely, we check that every element of $\mathcal{G}$ occurs as $\Gamma^*(I, J, \lambda, F)-$contribution. Since $U(R^+([1,n] \times I'), F) \triangleleft U_{\rho(I)}(F)$, and $U(R^+(J' \times [1,n]), F) \triangleleft U_{\rho(J')}(F)$, and since

$$U_{\rho(I)}(F) \times U_{\rho(J')}(F) \xrightarrow{\xi} U(R^+(I \times I), F) \times U(R^+(J \times J), F)$$

is a surjective homomorphism, where

$$(x_2x_1, y_2y_2) \mapsto (x_1^{\pi_1}, \quad y_1^{\pi_2})$$

the $\Gamma^*(I, J, \lambda, F)-$contributions generate $\Gamma^*(I, J, \lambda, F)$, and so

$$\Gamma^*(I, J, \lambda, F) = \Gamma(I, J, \lambda, F).$$

This is Theorem 1.

Theorems 2 and 3.

Set $\sum = \sum_{d-1}$, and set

$$\begin{aligned} E_1 &= (I \cap J)\pi_1, \quad F_1 = (I \cap J)\pi_2, \\ E_2 &= B\pi_1, \quad F_2 = A\pi_2, \\ E_3 &= D\pi_1, \quad F_3 = C\pi_2. \end{aligned} \tag{7.1}$$

Since $I = I \cap J \mathbin{\dot\cup} B \mathbin{\dot\cup} D$, and since π_1 agrees with $\lambda(I, [1,d])$ on I, it follows that

$$[1,d] = E_1 \mathbin{\dot\cup} E_2 \mathbin{\dot\cup} E_3. \tag{7.2}$$

Similarly

$$(7.3) \qquad [1,d] = F_1 \dot\cup F_2 \dot\cup F_3.$$

From (7.2) and (7.3), we conclude that $[1,d]$ is partitioned into nine sets $E_i \cap F_j, 1 \leq i,j \leq 3$, some of which may be empty. So

(7.4) $\qquad [1,d]^2$ is partitioned into 81 sets

$$E_i \cap F_j \times E_k \cap F_l, \quad 1 \leq i,j,k,l \leq 3.$$

These 81 sets are called <u>cells</u>, and much of the remaining discussion involves careful examination of these cells.

Using (1.33)-(1.38), we check that

$$(7.5) \qquad \begin{gathered} \text{if } (g_1,g_2),(h_1,h_2) \in \mathcal{G}, \text{ then} \\ ([g_1,h_1],[g_2,h_2]) \in \mathcal{G}; \end{gathered}$$

$$(7.6) \qquad \text{if } r,s \in \sum\nolimits^+, t \in F^\times, \text{ and } (x_s(t),x_r(t)) \in \mathcal{G},$$

$$\text{then } (x_s(t'),x_r(t')) \in \mathcal{G} \text{ for all } t' \in F;$$

$$(7.7) \qquad \text{if } r,r_1,s,s_1 \in \sum\nolimits^+, t,t_1 \in F^\times, \{r,s\} \cap \{r_1,s_1\} \neq \phi$$

and $(x_s(t),x_r(t)),(x_{s_1}(t_1),x_{r_1}(t_1)) \in \mathcal{G}$, then $r = r_1$ and $s = s_1$;

$$(7.8) \qquad \text{if } r,r_1,s \in \sum\nolimits^+, t,t_1 \in F^\times, \text{ and}$$

$$(x_s(t),x_r(t)),(1,x_{r_1}(t_1)) \in \mathcal{G}, \text{ then } r \neq r_1;$$

$$\text{(7.9)} \qquad \text{if } r, s, s_1 \in \sum\nolimits^+, t, t_1 \in F^\times, \text{ and}$$

$$(x_s(t), x_r(t)), (x_{s_1}(t_1), 1) \in \mathcal{G}, \text{ then } s \neq s_1.$$

Coupling (1.33)-(1.38) with (7.5)-(7.9), we see that there is an exact sequence

$$1 \to K_1 \to \Gamma(I, J, \lambda, F) \xrightarrow{p_1} L_1 \to 1,$$

where

$$K_1 = U(R^+([1, d] \times F_3), F), \tag{7.11}$$

$$L_1 = U(R^*, F), \tag{7.12}$$

$$\begin{aligned} R^* = {} & R^+(E_3 \times [1, d]) \cup R^+(E_1 \times E_1) \cup \\ & R^+(\{(a\lambda\pi_1, a'\lambda\pi_1) | a, a' \in A, a < a'\}) \cup \\ & R^+(\{(a\lambda\pi_1, j\pi_1) | a \in A, j \in I \cap J\}) \cup \\ & R^+(\{(j\pi_1, a\lambda\pi_1) | j \in I \cap J, a \in A, j < a\}). \end{aligned} \tag{7.13}$$

It is obvious that $R^+([1, d] \times F_3)$ is closed, and so is $R^+([1, d] \times F_3)' = R^+([1, d] \times (F_1 \cup F_2))$, so $K_1 = U_{\tau_1}(F)$ for some $\tau_1 \in S_d$. By (7.13), R^* is closed. To check that $R^{*'}$ is closed, we use the fact that

$$\begin{aligned} [1, d]^2 = {} & (E_1 \times E_1) \cup (E_1 \times E_2) \cup (E_1 \times E_3) \cup (E_2 \times E_1) \cup (E_2 \times E_2) \cup \\ & (E_2 \times E_3) \cup (E_3 \times E_1) \cup (E_3 \times E_2) \cup (E_3 \times E_3). \end{aligned}$$

Hence,

$$R^* = R^+(E_1 \times E_1) \cup R^+(\{(\alpha, \beta) \in E_1 \times E_2 | \alpha = j\pi_1, \beta = a\lambda\pi_1,$$

$$j \in I \cap J, \quad a \in A, \quad j < a\}) \cup \phi \cup$$

$$R^+(\{(\alpha, \beta) \in E_2 \times E_1 | \alpha = a\lambda\pi_1, \beta = j\pi_1, \quad a < j\}) \cup$$

$$R^+(\{(\alpha, \beta) \in E_2 \times E_2 | \alpha = a\lambda\pi_1, \beta = a'\lambda\pi, a, a' \in A, a < a'\}) \cup$$

$$\phi \cup R^+(E_3 \times E_1) \cup R^+(E_3 \times E_2) \cup R^+(E_3 \times E_3),$$

and so

$$\begin{aligned} R^{*'} =& \phi \cup R^+(\{(\alpha, \beta) \in E_1 \times E_2 | \alpha = j\pi_1, \beta = a\lambda\pi_1, j \in I \cap J, \quad a \in A, \quad j > a\} \cup \\ & R^+(E_1 \times E_3) \cup R^+(\{(\alpha, \beta) \in E_2 \times E_1 | \alpha = a\lambda\pi_1, \beta = j\pi_1, a > j\}) \cup \\ & R^+(\{(\alpha, \beta) \in E_2 \times E_2 | \alpha = a\lambda\pi_1, \beta = a'\lambda\pi_1, \quad a, a' \in A, a > a'\}) \cup \\ & R^+(E_2 \times E_3) \cup \phi \cup \phi \cup \phi, \end{aligned}$$

and we check that $R^{*'}$ is closed so that $L_1 = U_{\sigma_1}(F)$ for some $\sigma_1 \in S_d$. A similar argument produces τ_2 and σ_2. This is Theorem 2.

Theorem 3 will be shown to be a consequence of the following lemma.

Lemma. *Suppose $I, J \in P_d([1, n])$ and $I \cap J = \phi$. Let*

$$\begin{aligned} M_n(I, J, F) =& \{M \in M_n(F), M = (m_{ij}), \\ & m_{ij} = 0 \textit{ if } i \in I', \\ & m_{ij} = 0 \textit{ if } j \in J', \\ & m_{ij} = 0 \textit{ if } i > j\}. \end{aligned}$$

Let $P = U(R^+(I \times I), F), \quad Q = U(R^+(J \times J), F)$, Then $P \times Q$ acts on $M_n(I, J, F)$ via

$$M_n(I, J, F) \times (P \times Q) \to M_n(I, J, F),$$

$$(M, (P, Q)) \mapsto P^{-1}MQ.$$

Then $\{0\} \bigcup_{\lambda \in \Lambda(I,J)} M(I,J,\lambda,F)$ *is a set of representatives for the orbits of* $P \times Q$ *on* $M_n(I,J,F)$*, where*

$$M(I,J,\lambda,F) = \left\{ \sum_{a \in \mathcal{D}(\lambda)} t_a e_{a,a\lambda}, t_a \in F^\times \right\},$$

and λ *ranges over the nonempty maps in* $\Lambda(I,J)$*.*

The proof is omitted, being an exercise in row and column operations.

To be able to apply this lemma, we note that $U_{\rho(J')}(F) \times U_{\rho(I)}(F)$ contains

$$\delta(U(R^+(I' \cap J \times I' \cap J), F)) \times \delta(U(R^+(I \cap J' \times I \cap J'), F)).$$

Also, every orbit of $U_{\rho(J')}(F) \times U_{\rho(I)}(F)$ on $U_n(F)$ contains an element of $U(R_{\rho(J')\omega_0} \cap R_{\rho(I)\omega_0}, F)$. Since $R_{\rho(J')\omega_0} \cap R_{\rho(I)\omega_0} = R^+(I' \cap J \times I \cap J')$, we can bring the lemma into play by using the isomorphism

$$\begin{aligned} \iota : U(R^+(I' \cap J \times I \cap J'), F) \cong \\ M_n(I' \cap J, I \cap J', F) \\ \text{via } x_{\alpha,\beta}(t) \mapsto t e_{\alpha,\beta}, \end{aligned} \tag{7.14}$$

where $e_\alpha - e_\beta \in R^+(I' \cap J \times I \cap J')$. Also,

$$\begin{aligned} &\delta(U(R^+(I' \cap J \times I' \cap J), F)) \times \delta(U(R^+(I \cap J' \times I \cap J'), F)) \\ &\cong U(R^+(I' \cap J \times I' \cap J), F) \times U(R^+(I \cap J' \times I \cap J'), F) \end{aligned} \tag{7.15}$$

with the isomorphism being the deletion of δ, and one checks that (7.1) and (7.2) are compatible; conjugation action on $U(R^+(I' \cap J \times I \cap J'), F)$ by $\delta(U(R^+(I' \cap J \times I' \cap J), F))$ induces action on the left on $M_n(I' \cap J, I \cap J', F)$, and conjugation action on $U(R^+(I' \cap J \times I \cap J'), F)$ by $\delta(U(R^+(I \cap J' \times I \cap J'), F)$ induces action on the right on $M_n(I'\cap J, I\cap J', F)$. By the lemma, every orbit of $U_{\rho(J')}(F)\times U_{\rho(I)}(F)$ contains an element of the stated type.

Suppose $\lambda,\mu\in\Lambda(I,J), t_a\in F^\times$ for all $a\in\mathcal{D}(\lambda)$, $\quad u_a\in F^\times$ for all $a\in\mathcal{D}(\mu)$, and

$$\prod_{a\in\mathcal{D}(\lambda)} x_{a,a\lambda}(t_a), \prod_{a\in\mathcal{D}(\mu)} x_{a,a\mu}(u_a)$$

are in the same $\Gamma-$orbit. We must show that $\lambda=\mu$ and that $t_a=u_a$ for all $a\in\mathcal{D}(\lambda)$. Suppose false for $\lambda,\mu,t_a,a\in\mathcal{D}(\lambda),u_a,a\in\mathcal{D}(\mu)$. Set

$$R_1=\{e_a-e_{a\lambda}|a\in\mathcal{D}(\lambda)\},$$

$$R_2=\{e_a-e_{a\mu}|a\in\mathcal{D}(\mu)\},$$

$$R=R_1\cup R_2.$$

Let r_0 be the smallest root in the ($\underline{\rho}-$ordering) in R such that one of the following holds:

$(a)\quad r_0\notin R_1\cap R_2,$ and either $r_0=e_a-e_{a\lambda}\in R_1,$ with $t_a\neq 0,$ or $r_0=e_a-e_{a\mu}\in R_2$ with $u_a\neq 0$

$(b)\quad r_0\in R_1\cap R_2$ and $t_{a_0}\neq u_{a_0},$ where

$r_0=\quad e_{a_0}-e_{a_0\lambda},\quad a_0\in\mathcal{D}(\lambda)\cap\mathcal{D}(\mu)$ and

$a_0\lambda=a_0\mu.$

From the minimality of r_0, we conclude that if $s\underset{\rho}{<}r_0$ and $s\in R$, then $s\in R_1\cap R_2, s=e_a-e_{a\lambda}=e_a-e_{a\mu}, t_a=u_a$. Set

$$T_1=\prod_{a\in\mathcal{D}(\lambda)} x_{a,a\lambda}(t_a), T_2=\prod_{a\in\mathcal{D}(\mu)} x_{a,a\mu}(u_a),$$
$$T=\prod x_{a,a\lambda}(t_a),$$

where the product for T ranges over the $s=e_a-e_{a\lambda}\in\mathcal{D}(\lambda)$ with $s\underset{\rho}{<}r_0$, so that

$$T_1=T\cdot T(1), T_2=T\cdot T(2),$$

where, with no loss of generality, $r_0 = e_{a_0} - e_{a_0\lambda} \in R_1, a_0 \in \mathcal{D}(\lambda)$. Thus, $b_0 = a_0\lambda, a_0 \in I' \cap J, b_0 \in I \cap J', a_0 < b_0$. Also

$$T(1) = x_{a_0b_0}(t_{a_0}) \cdot \prod x_{a,a\lambda}(t_a),$$

$$T(2) = x_{a_0,b_0}(c) \cdot \prod x_{a,a\mu}(u_\mu),$$

where the product for $T(1)$ ranges over $r = e_a - e_{a\lambda} \in R_1$ with $r_0 \underset{\overline{\rho}}{\leq} r$, and the product for $T(2)$ ranges over $r = e_a - e_{a\mu} \in R_2$ with $r_0 \underset{\overline{\rho}}{\leq} r$, and where $c = 0$ if $r_0 \notin R_2, c = u_{a_0}$ if $r_0 \in R_2$. In all cases,

$$t_{a_0} \neq c.$$

At this point, I introduce the notion of retraction along a positive root r. Suppose $r = e_i - e_j \in \sum^+$. Set

$$\sum(i,j)^+ = \{s \in \sum^+ | s = e_{i'} - e_{j'} i \leq i' < j' \leq j\}$$

$$U(i,j,F) = \langle X_s(F) | s \in \sum(i,j)^+ \rangle.$$

Then there is an idempotent endomorphism φ_r of $U_d(F)$ which sends $x_s(t)$ to 1 if $s \notin \sum(i,j)^+$, and which fixes $x_s(t)$ if $s \in \sum(i,j)^+, t \in F$. The existence of φr is obvious, since $\sum(i,j)^+$ and $\sum(i,j)^{+'}$ are closed, and since $r_1 \in \sum^+, r_2 \in \sum(i,j)^{+'}$ and $r_1 + r_2 \in \sum^+$ imply that $r_1 + r_2 \in \sum(i,j)^{+'}$.

We consider $\varphi = \varphi_{r_0}$ and observe that

$$\varphi(U_{\rho(J')}(F)) \subseteq U_{\rho(J')}(F),$$

$$\varphi(U_{\rho(I)}(F)) \subseteq U_{\rho(I)}(F).$$

Set

$$P_1 = \varphi(T_1), \quad P_2 = \varphi(T_2), \quad P = \varphi(T),$$

so that

$$P_1 = Px_{r_0}(t_{a_0}), P_2 = Px_{r_0}(c),$$

where one of the following holds:

(i) $c = 0$ and $r_0 \notin R_2$.

(ii) $c = u_a \neq t_a$.

By hypothesis, there are $g_1 \in U_{\rho(J')}(F), g_2 \in U_{\rho(I)}(F)$, such that $g_1 T_1 = T_2 g_2$.

Set

$$g = \varphi(g_1), h = \varphi(g_2),$$

so that

(*) $$gPx_{r_0}(t_{a_0}) = Px_{r_0}(c)h.$$

Set

$$U_0 = U(a_0, b_0, F); U_1 = U(R^*, F),$$

where $R^* = \{e_{i'} - e_{j'} | a_0 < i' < j' < b_0\}$ Thus $P \in U_1$, and

$$U_0 = U_1 U_2, U_1 \cap U_2 = \{1\},$$

where

$$U_2 = U(R^{**}, F),$$

and

$$R^{**} = \{e_{a_0} - e_j | a_0 < j \leq b_0\} \cup \{e_i - a_{b_0} | a_0 < i < b_0\}.$$

Note that

$$U_2 \lhd U_0.$$

We first treat a special case. Suppose $P = 1$. Here (*) becomes

$$gx_{r_0}(t_{a_0}) = x_{r_0}(c)h.$$

Since $X_{r_0}(F)$ is in the center of U_0, we get

$$gh^{-1} = x_{r_0}(c - t_{a_0}) = x_{r_0}(b), b \in F^{\times}.$$

Equivalently, $g = x_{r_0}(b)h$. This implies that either $r_0 \in R_{\rho(J')}$ or $r_0 \in R_{\rho(I)}$. Since $a_0 \in I' \cap J, b_0 \in I \cap J'$, we get $r_0 \notin R_{\rho(J')}, r_0 \notin R_{\rho(I)}$. We conclude that

$$P \neq 1.$$

Since $P \in U_1$, this forces $a_0 + 1 < b_0$, and so

$$[U_2, U_2] = X_{r_0}(F).$$

Write

$$g = \tilde{g}u, \quad h = \tilde{h}\upsilon,$$

where $\tilde{g}, \tilde{h} \in U_1, \quad u, \upsilon \in U_2$. From (*), we get

$$\tilde{g}uP = P\tilde{h}\upsilon x_{r_0}(b).$$

Since $P \in U_1$, this gives us two equations:

$$\tilde{g}P = P\tilde{h}, u^P = \upsilon x_{r_0}(b),$$

$$u \in U_{\rho(J')}(F) \cap U_1, \quad \upsilon \in U_{\rho(I)}(F) \cap U_1.$$

Write

$$u = u_1u_2, \upsilon = \upsilon_1\upsilon_2,$$

where

$$u_1 = \prod_{j=a_0+1}^{b_0-1} x_{a_0 j}(f_j), \quad u_2 = \prod_{j=a_0+1}^{b_0-1} x_{j b_0}(f'_j),$$

$$\upsilon_1 = \prod_{j=a_0+1}^{b_0-1} x_{a_0 j}(e_j), \quad \upsilon_2 = \prod_{j=a_0+1}^{b_0-1} x_{j b_0}(e'_j).$$

Here I am using $r_0 \notin R_{\rho(J')}, r_0 \notin R_{\rho(I)}$.

Since P normalizes $\langle X_{a_0 j}(F) | a_0 < j < b_0 \rangle$ and $\langle X_{j b_0}(F) | a_0, < j < b_0 \rangle$, it follows that

$$u^P = \tilde{u}_1 \tilde{u}_2,$$

where

$$\tilde{u}_1 = \prod_{j=a_0+1}^{b_0-1} x_{a_0 j}(\tilde{f}_j), \quad \tilde{u}_2 = \prod_{j=a_0+1}^{b_0-1} x_{j b_0}(\tilde{f}'_j),$$

whence from $u^P = \upsilon x_{r_0}(b)$, we get $b = 0$. This contradiction shows that Theorem 3 holds.

8. Partitions and associated groups

In order to study the groups $\Gamma(I, J, \lambda, F)$, I introduce a graph $\Gamma = \Gamma(I, J, \lambda)$.

I set

$$V(\Gamma) = [1,d],$$

$$E(\Gamma) = \{(\mu,\nu) | \text{ one of the following holds;}$$

(8.0) $\qquad$ (1) $\quad j_\mu = i_\nu,$

(2) $\quad j_\mu \in A, \quad i_\nu \in B$ and

$j_\mu \lambda = i_\nu \}.$

Here I am using (1.27) and (1.32).

From (3.8), together with the definitions of $\pi(I)$ and $\pi(J)$, we get

(8.1) $\qquad i_\mu < j_{\mu\omega}$ for all $\mu \in [1, d]$.

Lemma 8.1. *$i_\mu < j_\mu$ for all $\mu \in [1, d]$.*

Proof. Fix $\mu \in [1, d]$. Since

$$|\{\nu\omega | \nu \in [\mu, d]\}| = \ d - \mu + 1 \text{ and } |[1,\ \mu]| = \mu,$$

It follows that

$$\{\nu\omega | \nu \in [\mu, d]\} \cap [1, \mu] \neq \phi.$$

Choose $k \in \{\nu\omega | \nu \in [\mu, d]\} \cap [1, \mu]$. Thus

$$k = \kappa\omega \leq \mu \text{ for some } \kappa \in [\mu, d].$$

It follows that

$$i_\mu \leq i_\kappa < j_{\kappa\omega} \leq j_\mu,$$

and the lemma follows.

Lemma 8.2. *If $(\mu, \nu) \in E(\gamma)$, then $\mu < \nu$.*

Proof. If (1) holds for (μ, ν), in (8.0), then $j_\mu = i_\nu$. By Lemma 8.1, $i_\nu < j_\nu$, and so $j_\mu < j_\nu$, whence $\mu < \nu$. If (2) holds for (μ, ν) in (8.0), then

$$j_\mu \in A, i_\nu \in B \text{ and } j_\mu\lambda = i_\nu.$$

By definition of λ in (1.30 ii) we get

$$j_\mu < i_\nu.$$

By Lemma 8.1, $i_\nu < j_\nu$, and so $j_\mu < j_\nu$, whence $\mu < \nu$.

Lemma 8.3. *If (μ, ν_1) and (μ, ν_2) are edges of Γ, then $\nu_1 = \nu_2$.*

Proof. Since $A \subseteq I'$, if follows that if (1) holds for (μ, ν_i), then (1) also holds for (μ, ν_{3-i}). The lemma follows.

Lemma 8.4. *If (μ_1, ν) and (μ_2, ν) are edges of Γ, then $\mu_1 = \mu_2$.*

Proof. Since $B \subseteq J'$, it follows that if (1) holds for (μ_i, ν), then (1) holds for (μ_{3-i}, ν). The lemma follows.

It follows from Lemmas 8.1, 8.2 and 8.3 that if Γ' is a connected component if Γ, then

$$V(\Gamma') = \{a_1, a_2, \dots, a_l\} \quad a_1 < a_2 < \cdots < a_l,$$

and

$$E(\Gamma') = \{(a_i, a_{i+1})| \quad i \in [1, l-1]\}.$$

Let $\Gamma_1, \dots, \Gamma_k$ be the connected components of Γ, ordered so that $|V(\Gamma_i)| = \mu_i$ and

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k.$$

Set $\mu(I, J, \lambda) = \mu = (\mu_1, \dots, \mu_k)$,

$$V(\Gamma_i) = \{a_{1i}, a_{2i}, \dots, a_{\mu_i i}\} \quad a_{1i} < a_{2i} < \cdots < a_{\mu_i i}, \tag{8.2}$$

$$D(\mu) = \{(x, y) \in \mathbb{N}^2 | y \in [1, k], x \in [1, \mu_y]\}. \tag{8.3}$$

I call $D(\mu)$ the dot diagram of μ, and define

$$\varphi(I, J, \lambda) = \varphi : D(\mu) \to [1, d] \tag{8.4}$$

$$\varphi(x, y) = a_{xy}.$$

Since $[1, d]$ is the disjoint union of the $V(\Gamma_i)$, it follows that φ is a bijection.

From the map φ, I construct a group $G(\varphi, F)$ for each field F, and as with $G(I, J, \lambda, F)$, I give $G(\varphi, F)$ by giving a set of generators. As with $G(I, J, \lambda, F)$, $G(\varphi, F)$ is a subgroup of $U_d(F) \times U_d(F)$. Here are generators:

$$\begin{aligned} &\{(x_{ij}(t), 1) | t \in F, i < j, \\ &\quad i = \varphi(1, y) \text{ for some } y \in [1, k]\}, \end{aligned} \tag{8.5}$$

$$\begin{aligned} &\{(1, x_{ij}(t) | t \in F, i < j, \\ &\quad j = \varphi(\mu_y, y) \text{ for some } y \in [1, h]\}, \end{aligned} \tag{8.6}$$

$$\begin{aligned} &\{(x_{i',j'}(t), x_{ij}(t)) | t \in F, \quad i < j, \quad i' < j' \\ &i = \varphi(a, y), \quad j = \varphi(b, z), \\ &i' = \varphi(a + 1, y), j' = \varphi(h + 1, z), \\ &\text{for some } \{(a, y), (a + 1, y), (b, z), (b + 1, z)\} \subseteq D(\mu)\}. \end{aligned} \tag{8.7}$$

Theorem 8.1. *For all (I, J, λ), and all fields F*

$$G(I, J, \lambda, F) = G(\varphi, F),$$

where

$$\varphi = \varphi(I, J, F).$$

Proof. We prove that each generator in any of (8.5), (8.6), (8.7) occurs in one of (1.33)-(1.39), and conversely, that each of the elements appearing in (1.33)-(1.39) is contained in $G(\varphi, F)$.

I start with (8.5). Suppose $t \in F, \quad 1 \leq \mu < \nu \leq d$ and $\mu = \varphi(1, y)$ for some μ. From the definition of Γ, this implies that $i_\mu \in J'$ and in addition $i_\mu \notin B$. Thus, $i_\mu \in I \cap J' - B$, that is $i_\mu \in D$. By (1.32), $\pi_1 = \pi(I)$, and by definition of $\pi(I)$, we get

$$i_\mu \pi_1 = \mu.$$

Since $\mu < \nu$, it follows that $e_\mu - e_\nu \in R^+(D\pi_1 \times [1, d])$, and so

$$(x_{\mu\nu}(t), 1)$$

is one of the elements appearing in (1.33). The remaining assertions follow in a similar manner.

Let $\varphi = \varphi(I, J, \lambda)$ as before, and set

$$\Omega(\varphi) = \{\omega \in S_d | \text{ if } \{(a, l), (a + 1), l)\} \subseteq D(\mu),$$

$$\text{then } [\varphi(a + 1, l), d]\omega \subseteq [\varphi(a, l) + 1, d]\}.$$

Lemma 8.5. *If* $\omega \in S_d$ *and*

$$i_\sigma < j_{\sigma\omega} \text{ for all } \sigma \in [1, d],$$

then $\omega \in \Omega(\varphi)$.

Proof. Suppose $i_\sigma < j_{\sigma\omega}$ for all $\sigma \in [1, d]$ and $\omega \notin \Omega(\varphi)$. Then for some $\{(a, l), (a + 1, l)\} \subseteq D(\mu)$ and some $\nu' \geq \varphi(a + 1, l)$, we have

$$\nu'\omega \leq \varphi(a, l).$$

Set

$$\sigma = \varphi(a, l), \quad \sigma' = \varphi(a+1, l).$$

By definition of Γ, one of the following holds:

$$(a) \quad j_\sigma = i_{\sigma'}.$$

$$(b) \quad j_\sigma \in A, \iota_{\sigma'} \in B \text{ and } j_\sigma \lambda = i_{\sigma'}.$$

In either case, we have

$$j_\sigma \leq i_{\sigma'}, \text{ and } \sigma < \sigma'.$$

Namely, $i_{\sigma'} < j_{\sigma'}$, by Lemma 8.2. Thus, if (a) holds, then $j_\sigma = i_{\sigma'}, < j_{\sigma'}$, so that $\sigma < \sigma'$. If (b) holds, then since λ is increasing, we get $j_\sigma < i_{\sigma'}$, and as $i_{\sigma'} < j_{\sigma'}$ by Lemma 8.2, we get $j_\sigma < j_{\sigma'}$ so $\sigma < \sigma'$.

We are given $\varphi(a+1, l) = \sigma'$ and $\nu' \geq \sigma', \nu'\omega \leq \sigma$.

First, suppose that $j_\sigma = i_{\sigma'}$. Since $\nu' \in [1, d]$, we have $j_\sigma = i_{\sigma'} \leq i_{\nu'} < j_{\nu'\omega} \leq j_\sigma$, a contradiction.

Suppose that $j_\sigma \lambda = i_{\sigma'}$, so that $j_\sigma < i_{\sigma'}$; now we get

$$j_\sigma < i_{\sigma'} \leq i_{\nu'} < j_{\nu'\omega} \leq j_\sigma,$$

a contradiction.

9. Constructing some (I, J, λ) from labeled dot diagrams

In this section, I start with a partition μ of d:

$$\mu = (\mu_1, \ldots, \mu_k).$$

Set

$$D(\mu) = \{(x, y) \in \mathbb{N}^2 | y \in [1, k], x \in [1, \mu_y]\}, \tag{9.1}$$

the dot diagram of μ. Set

$$\Phi(\mu) = \{\varphi : D(\mu) \to [1,d] | \ \varphi \text{ is a bijection and}$$

(9.2)

$$\varphi(x,y) < \varphi(x+1,y) \text{ for all } \{(x,y),(x+1,y)\} \subseteq D(\mu)\}.$$

If $\varphi \in \Phi(\mu)$ and F is a field, set

$$G_L = \{x_{ij}(t),1) | t \in F, i < j, \text{ and } i = \varphi(1,y)$$

$$\text{for some } y \in [1,k]\},$$

$$G_R = \{(1, x_{ij}(t)) | t \in F, i < j \text{ and } j = \varphi(\mu_y, y)$$

(9.3)

$$\text{for some } y \in [1,k]\},$$

$$G_D = \{x_{i'j'}(t), x_{ij}(t)) | t \in F, i < j, i' < j',$$

$$\text{and}$$

$$i = \varphi(a,y_1), j = \varphi(b,y_2),$$

$$i' = \varphi(a+1,y_1), j' = \varphi(b+1,y_2)$$

$$\text{for some } \{(a,y_1),(b,y_2),(a+1,y_1),(b+1,y_2)\} \subseteq D(\mu)\}$$

$G(\varphi, F) =< G_L \cup G_R \cup G_D >$.

Theorem 9.1. *If $\mu \vdash d$ and $\varphi \in \Phi(\mu)$, then for some $n \in \mathbb{N}$, there are subsets $I, J \subseteq [1,n], |I| = |J| = d$, subsets $A \subseteq I' \cap J, B \subseteq I \cap J'$ and a map $\lambda : A \to B$ such that λ is a bijection and $a < a\lambda$ for all $a \in A$, such that for all fields F,*

$$G(I,J,\lambda,F) = G(\varphi,F).$$

In addition, setting

$$\Omega(\varphi) = \{\omega \in S_d | [\varphi(a+1,l),d]\omega \subseteq [\varphi(a,\ell)+1,d]$$

$$\textit{for all } \{(a,l),(a+1,l)\} \subseteq D(\mu)\},$$

I, J have the property that

$$I = \{i_1, i_2, \dots, i_d\}, \quad i_1 < i_2 < \cdots < i_d,$$

$$J = \{j_1, j_2 \dots, j_d\} \quad j_1 < j_2 < \cdots < j_d,$$

and

$$\Omega(\varphi) = \{\omega \in S_d | i_\sigma < j_{\sigma\omega} \textit{ for all } \sigma \in [1, d]\}.$$

Proof. Set

$$\mathcal{X}(\varphi) = \{(I, J, \lambda) | \quad \varphi(I, J, \lambda) = \varphi\}.$$

Among other things, we must prove that $\mathcal{X}(\varphi) \neq \phi$ for all $\varphi \in \Phi(\mu)$. I remark that if $(I, J, \lambda) \in \mathcal{X}(\varphi)$, then by Lemma 8.5, it follows that

$$\{\omega \in S_d | i_\sigma < j_{\sigma\omega} \text{ for all } \sigma \in [1, d]\} \subseteq \Omega(\varphi).$$

Set

$$L(\varphi) := \{(\varphi(a, \alpha), \varphi(a + 1, \alpha)) | \quad \{(a, \alpha), (a + 1, \alpha)\} \subseteq D(\mu)\},$$

$$\Lambda(\varphi) := \{(x, y) \in L(\varphi)^2 | x = (i, j), y = (i', j')$$

$$\text{and } i' < i < j < j'\}.$$

If $\{x, y\} \subseteq L(\varphi)$, set $x <_\varphi y$ if $(x, y) \in \Lambda(\varphi)$. By inspection, $(L(\varphi), <_\varphi)$ is a poset.

Set

$L_{\min}(\varphi) := \{x \in L(\varphi) | x \text{ is minimal under } <_\varphi\}$.

Set

$$l := |L_{\min}(\varphi)|. \tag{9.1}$$

Since φ is a bijection, it follows that if

$$\{(i,j),(i',j')\} \subseteq L(\varphi) \text{ and } (i,j) \neq (i',j'), \text{ then } i \neq i' \text{ and } j \neq j'. \tag{9.2}$$

From (9.1), it follows that

$$L_{\min}(\varphi) = \{(\mu_1, \nu_1), \ldots, (\mu_l, \nu_\ell)\},$$

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_\ell.$$

Since $(\mu_h, \nu_h) \in L(\varphi)$ for all $h \in [1,\ell]$ and $\varphi \in \Phi(\mu)$, it follows that

$$\mu_h < \nu_h \quad \forall h \in [1,\ell]. \tag{9.3}$$

It follows from (9.2) that

$$\mu_1 < \mu_2 < \cdots < \mu_\ell, \tag{9.4}$$

and it also follows from (9.2) that $\nu_h \neq \nu_{h+1} \quad \forall h \in [1,\ell-1]$.

Suppose by way of contradiction that $\nu_h > \nu_{h+1}$ for some $h \in [1,\ell-1]$. Then (9.3) and (9.4) yield $\mu_h < \mu_{h+1} < \nu_{h+1} < \nu_h$, whence $(\mu_{h+1}, \nu_{h+1}) <_\varphi (\mu_h, \nu_h)$, against $(\mu_h, \nu_h) \in L_{\min}(\varphi)$. So

$$\nu_1 < \nu_2 < \cdots < \nu_\ell. \tag{9.5}$$

Set

$$\mu_0 := 0, \mu_{\ell+1} := d, \nu_{\ell+1} : d+1, \tag{9.6}$$

and set

$$\begin{aligned} c_{2h-1} &:= \nu_h - h + \mu_{h-1}, \\ c_{2h} &:= \nu_h - h + \mu_h \quad \forall h \in [1,\ell+1]. \end{aligned} \tag{9.7}$$

Set

$$I := [1, c_1] \cup \bigcup_{h \in [1,\ell]} [c_{2h}, c_{2h+1}], \tag{9.8}$$

$$J := \bigcup_{h \in [1,\ell+1]} [c_{2h-1} + 1, c_{2h}]. \tag{9.9}$$

Lemma 9.1.

$$(i) \quad c_{2h-1} < c_{2h} \quad \forall h \in [1, \ell].$$

$$(ii) \quad c_{2\ell+1} < c_{2\ell+2}.$$

$$(iii) \quad c_{2h} \le c_{2h+1} \quad \forall h \in [1, \ell].$$

Proof. From (9.4) and (9.6), $\mu_{h-1} < \mu_h \quad \forall h \in [1, \ell - 1]$, so $\nu_h - h + \mu_{h-1} < \nu_h - h + \mu_h$. This is (i). Since $\mu_\ell < \nu_\ell \le d$, we get $\nu_{\ell+1} - (\ell+1) + \mu_\ell < \nu_{\ell+1} - (\ell+1) + d$. This is (ii).

If $h \in [1, \ell - 1]$, then $\nu_h < \nu_{h+1}$ by (9.5), so $\nu_h - h < \nu_{h+1} - h$ and so $\nu_h - h \le \nu_{h+1} - (h + 1)$, so

$$\nu_h - h + \mu_h \le \nu_{h+1} - (h + 1) + \mu_h.$$

So (iii) holds for all $h \in [1, \ell - 1]$. Since $2\ell + 1 = 2(\ell + 1) - 1$, and $\nu_\ell < \nu_{\ell+1}$, we get $\nu_\ell - \ell + \mu_\ell \le \nu_{\ell+1} - (\ell + 1) + \mu_\ell$, so (iii) holds.

Lemma 9.2.

$$(i) \quad c_m \le c_{m+1}, \quad \forall m \in [1, 2\ell + 1].$$

(ii) If $h \in [1, \ell - 1]$, then

$$c_{2h} = c_{2h+1} \iff \nu_{h+1} = \nu_h + 1.$$

$$(iii) \quad c_{2\ell} = c_{2\ell+1} \iff \nu_\ell = d.$$

$$(iv) \quad c_{2h} < c_{2(h+1)}, \quad \forall h \in [1, \ell].$$

Proof. Lemma 9.1 implies (i). If $h \in [1, \ell - 1]$, then by (i) $c_{2h+1} - c_{2h} \geq 0$. Since

$$c_{2h+1} - c_{2h} = \nu_{h+1} - \nu_h - 1,$$

(ii) follows. Since $c_{2\ell+1} - c_{2\ell} = d - \nu_\ell$, (iii) follows.

Since $\mu_h < \mu_{h+1}$ and $\nu_h < \nu_{h+1} \quad \forall h \in [1, \ell]$, (iv) follows.

Lemma 9.3. $|I| = d$.

Proof. Since $0 < \mu_1 < \nu_1$, it follows that $\nu_1 > 1$, so $c_1 \geq 1$.

It follows from Lemma 9.1 and (9.8) that

$$\begin{aligned} |I| =& c_1 + \sum_{h \in [1,\ell]} (c_{2h+1} - c_{2h} + 1) \\ &= \ell + c_1 + \sum_{h \in [1,\ell]} (c_{2h+1} - c_{2h}), \end{aligned}$$

so this lemma follows from (9.6) and (9.7).

Lemma 9.4. $|J| = d$.

Proof. Since $c_{2h-1} \leq c_{2h}, \quad \forall h \in [1, \ell + 1]$, it follows that

$$|[c_{2h-1} + 1, c_{2h}]| = c_{2h} - c_{2h-1}.$$

Hence, by (9.9),

$$|J| = \sum_{h \in [1,\ell+1]} (c_{2h} - c_{2h-1}),$$

so this lemma follows from (9.7).

Remarks. For future reference, I note that $c_{2\ell+2} = 2d - \ell$.

Lemma 9.5. $(i) I \cup J \supseteq [1, 2d - \ell]$.

$(ii) I \cap J \supseteq \{c_{2h} | h \in [1, \ell]\}$.

Proof. Set $X := I \cup J$. By construction, $1 \in X$. Suppose $m \in X$ and $m < 2d - \ell$. Then m is in one of the following intervals:

$(1)[1, c_1]$.

$(2)[c_{2h}, c_{2h+1}], h \in [1, \ell]$.

$(3)[c_{2h-1} + 1, c_{2h}], h \in [1, \ell]$.

$(4)[c_{2\ell+1}, c_{2\ell+2} - 1]$.

In each case, inspection shows that $m + 1 \in X$, so by finite induction, (i) holds. By (9.8) and (9.9), (ii) holds.

Lemma 9.6.

$(i) \quad I \cup J = [1, 2d - \ell]$.

$(ii) \quad I \cap J = \{c_{2h} | h \in [1, \ell]\}$.

Proof. By Boolean algebra

$$I \cup J = (I \setminus I \cap J) \cup (J \setminus I \cap J) \cup I \cap J$$

and so

$$|I \cup J| = |I| + |J| - |I \cap J|.$$

By Lemmas (9.3)-(9.5), $|I \cup J| \geq 2d - \ell$ and $|I \cap J| \geq \ell,$ so

$$2d = |I| + |J| = |I \cap J| + |I \cup J| \geq \ell + 2d - \ell.$$

Thus, the inequality is an equality, and the lemma follows.

Set

$$i_\mu = \mu\lambda([1, d], I), \quad j_\mu = \mu\lambda([1, d], J), \forall \mu \in [1, d].$$

Lemma 9.7.

$$(i) \quad i_{\nu_h} = c_{2h},$$

$$(ii) \quad j_{\mu_h} = c_{2h}, \quad \forall h \in [1, \ell].$$

Proof. If $h \in [1, \ell]$, set

$$\overline{\nu}_h = |I \cap [1, c_{2h}]|.$$

Obviously, (9.8) implies that

$$c_{2h} = \max I \cap [1, c_{2h}], \tag{9.10}$$

so $c_{2h} = i_{\overline{\nu}_h}$.

From (9.8), we get

$$I \cap [1, c_{2h}] = \begin{cases} \{c_{2h}\} \cup [1, c_1] \text{ if } h = 1 \\ \{c_{2h}\} \cup [1, c_1] \cup \bigcup\limits_{k \in [1, h-1]} [c_{2k}, c_{2k+1}] \text{ if } h > 1. \end{cases}$$

So

$$|I \cap [1, c_{2h}]| = \begin{cases} \nu_1 \text{ if } h = 1 \\ \\ \nu_1 + \sum\limits_{k \in [1, h-1]} (c_{2k+1} - c_{2k} + 1) \text{ if } h > 1. \end{cases}$$

Since $c_{2k+1} - c_{2k} + 1 = \nu_{k+1} - (k+1) + \mu_k - \nu_k + k - \mu_k + 1 = \nu_{k+1} - \nu_k$, we get

$$\overline{\nu}_h = \nu_h, \forall h \in [1, \ell].$$

This is (i).

From (9.9), we get

$$|J \cap [1, c_{2h}]| = \sum_{k \in [1, h]} (c_{2k} - c_{2k-1}).$$

Since $c_{2k} - c_{2k-1} = \nu_k - k + \mu_k - \nu_k + k - \mu_{k-1} = \mu_k - \mu_{k-1}$, and since $\mu_0 = 0$, we get

$$|J \cap [1, c_{2h}]| = \mu_h, c_{2h} = \max J \cap [1, c_{2h}],$$

and (ii) follows.

Lemma 9.8. *If $m \in [\nu_\ell + 1, d]$, then*

$$i_m < j_{\mu_\ell + 1}.$$

Proof. Since $\nu_\ell < m$, it follows from Lemma 9.7 that

$$c_{2\ell} = i_{\nu_\ell} < i_m. \tag{9.11}$$

From (9.11) and (9.8), we get $i_m \in [c_{2\ell} + 1, c_{2\ell+1}]$.

Since $j_{\mu_\ell+1} > j_{\mu_\ell} = c_{2\ell}$, (9.9) implies that $j_{\mu_\ell+1} \in [c_{2\ell+1} + 1, c_{2\ell+2}]$. Since $c_{2\ell+1} < c_{2\ell+1} + 1$, the lemma follows.

Lemma 9.9. *If $m \in [1, \nu_1 - 1]$, then $i_m < j_1$.*

Proof. This is obvious.

Lemma 9.10. *If $\nu_h < m < \nu_{h+1}$ and $h \in [1, \ell - 1]$, then $i_m < j_{\mu_h+1}$.*

Proof. By Lemma 9.7, $i_{\nu_h} = c_{2h} < i_m < c_{2h+2}$.

Hence,

$$i_m \in [c_{2h}, c_{2h+1}]. \tag{9.12}$$

By Lemma 9.7, together with $\mu_h < \nu_h$, it follows that

$$c_{2h} < j_{\mu_h+1} \leq j_{\mu_{h+1}} = c_{2h+2}, \tag{9.13}$$

and so

$$j_{\mu_h+1} \in [c_{2h+1} + 1, c_{2h+2}]. \tag{9.14}$$

The lemma follows from (9.12), (9.14).

If $h \in [1, \ell]$, there is $(a_h, \alpha_h) \in D(\mu)$ such that

$$\mu_h = \varphi(a_h, \alpha_h), \nu_h = \varphi(a_h + 1, \alpha_h).$$

Set

$$X(\varphi) := \{(a_h, \alpha_h) | h \in [1, \ell]\},$$

$$Y(\varphi) := D(\mu) \setminus X(\varphi).$$

Let

$$Z(\varphi) := \{(a, \alpha) \in Y(\varphi) | (a + 1, \alpha) \in D(\mu)\}.$$

By construction, if $(a, \alpha) \in Z(\varphi)$, then there is $h \in [1, \ell]$ such that

$$\varphi(a, \alpha) < \varphi(a_h, \alpha_h) < \varphi(a_h + 1, \alpha_h) < \varphi(a + 1, \alpha). \tag{9.15}$$

Set

$$A(\varphi) := \{j_{\varphi(a,\alpha)} | (a, \alpha) \in Z(\varphi)\}$$

$$B(\varphi) := \{i_{\varphi(a+1,\alpha)} | (a, \alpha) \in Z(\varphi)\},$$

and define λ by

$$\lambda : A(\varphi) \to B(\varphi)$$

$$j_{\varphi(a,\alpha)} \mapsto j_{\varphi(a,\alpha)}\lambda = i_{\varphi(a+1,\alpha)}.$$

Obviously λ is a bijection.

It follows from (9.15) that

$$j_{\varphi(a,\alpha)} < j_{\varphi(a_h,\alpha_h)} = i_{\varphi(a_h+1,\alpha_h)} < i_{\varphi(a+1,\alpha)},$$

so

$$a < a\lambda \text{ all } a \in A(\varphi). \tag{9.16}$$

Putting these pieces together, it follows that

$$\varphi(I, J, \lambda) = \varphi,$$

that is, $(I, J, \lambda) \in \mathcal{X}(\varphi)$.

Set

$$\begin{aligned}\Omega^*(\varphi) := \{\omega \in S_d | \\ &[\varphi(a+1, \alpha), d]\omega \subseteq [\varphi(a, \alpha) + 1, d] \\ &\text{for all } \{(a, \alpha), (a+1, \alpha)\} \subseteq D(\mu)\}.\end{aligned}$$

It remains to prove that

$$i_m < j_{m\omega} \quad \forall \omega \in \Omega^*(\varphi), \text{ and } \forall m \in [1, d].$$

Set

$$\Omega^{**}(\varphi) = \{\omega \in S_d | [\nu_h, d]\omega \subseteq [\mu_h + 1, d] \quad \forall h \in [1, \ell]\}.$$

If $h \in [1, \ell]$, then

$$\mu_h = \varphi(a_h, \alpha_h), \nu_h = \varphi(a_h + 1, \alpha_h).$$

Hence

$$\Omega^*(\varphi) \subseteq \Omega^{**}(\varphi).$$

Conversely, suppose $\omega \in \Omega^{**}(\varphi)$ and $\{(a, \alpha), (a+1, \alpha)\} \subseteq D(\mu)$.

I argue that

$$[\varphi(a+1, \alpha), d]\omega \subseteq [\varphi(a, \alpha) + 1, d]. \tag{9.17}$$

If $(\varphi(a, \alpha), \varphi(a+1, \alpha)) \in \Lambda_{\min}(\varphi)$, then for some $h \in [1, \ell], \varphi(a, \alpha) = \mu_h$, $\varphi(a+1, \alpha) = \nu_h$ and (17) holds. If $(\varphi(a, \alpha), \varphi(a+1, \alpha)) \neq \Lambda_{\min}(\varphi)$, then there is

$h \in [1, \ell]$ such that

$$\varphi(a, \alpha) < \varphi(a_h, \alpha_h) < \varphi(a_h + 1, \alpha_h) < \varphi(a + 1, \alpha),$$

and so

$$\begin{aligned}
[\varphi(a+1, \alpha), d]\omega &\subseteq [\varphi(a_h + 1, \alpha_h), d]\omega \\
&\subseteq [\varphi(a_h, \alpha_h) + 1, d] \\
&\subseteq [\varphi(a, \alpha) + 1, d],
\end{aligned}$$

so (9.17) holds, whence $\Omega^{**}(\varphi) \subseteq \Omega^*(\varphi)$; so

$$\Omega^*(\varphi) = \Omega^{**}(\varphi).$$

Pick $m \in [1, d]$. Then one of the following holds:

1. $\nu_\ell < m$.
2. $m < \nu_1$.
3. $\nu_h < m < \nu_{h+1}$ for some $h \in [1, \ell - 1]$.
4. $m \in \{\nu_1, \nu_2, \ldots, \nu_\ell\}$.

Suppose $\nu_\ell < m$. Then $m\omega \geq \mu_\ell + 1$, as $\omega \in \Omega^{**}(\varphi)$.

By Lemma 7, $i_m < j_{\mu_\ell + 1} \leq j_{m\omega}$.

Similarly, if $m < \nu_1$, Lemma 9.9 applies, and if 3 holds, Lemma 9.10 applies. Finally, suppose $m = \nu_h$. Then $i_m = i_{\nu_h} = j_{\mu_h} < j_{\mu_h + 1} \leq j_{m\omega}$. The proof of the Theorem is complete.

10. The partitions (1^d) and (d)

Although I have been unsuccessful in proving that for all $\mu \vdash d, \varphi \in \Phi(\mu), \quad \omega \in \Omega(\varphi)$, there are polynomials $f(\mu, \varphi, \omega, \lambda) \in \mathbb{Z}[x]$ such that for all finite fields $\mathbb{F}_q$

$$|U_d(\mathbb{F}_q)\omega U_d(\mathbb{F}_q)/G(\varphi, \mathbb{F}_q)| = f(\mu, \varphi, \omega, q), \tag{10.1}$$

I have proved this assertion for two particular partitions μ of d. Namely, in this final section, I prove that (10.1) holds if

$$\mu \in \{(1^d), (d)\}$$

Case 1. $\mu = (1^d)$.

In this case, $\Phi(\mu)$ is the set of all bijections from $D(\mu)$ to $[1, d]$, and $\Omega(\varphi) = S_d$. On the other hand, $< G_L >= U_d(F) \times 1$ and

$$< G_R >= 1 \times U_d(F), \text{ so that}$$

$$G(\varphi, F) = U_d(F) \times U_d(F),$$

whence

$$|U_d(\mathbb{F}_q)\omega U_a(\mathbb{F}_q)/G(\varphi, \mathbb{F}_q)| = 1,$$

so we take $f(\mu, \varphi, \omega, \lambda) = 1$, and (10.1) holds.

Case 2. $\mu = (d)$.

In this case

$$D(\mu) = \{(x, 1) | 1 \le x \le d\},$$

and

$$|\Phi(\mu)| = 1.$$

The unique φ in $\Phi(\mu)$ is defined by

$$\varphi(x, 1) = x, \quad \forall x \in [1, d].$$

In this situation

$$< G_L >=< x_{1i}(t) | 2 \le i \le d, t \in F >,$$

$$< G_R >=< x_{id}(t) | 1 \le i \le d-1, t \in F >,$$

and

$$< G_D >=< (x_{i+1,j+1}(t), x_{i,j}(t)) | t \in F,$$

$$1 \leq i \leq j \leq d-1 > .$$

As for $\Omega(\varphi)$, we see that if $\omega \in \Omega(\varphi)$, then for each $i \in [1, d-1]$,

$$[i+1, d]\omega \subseteq [i+1, d],$$

and so

$$\Omega(\varphi) = \{1\}.$$

From the structure of $G(\varphi, F)$, it follows trivially that

$$|U_d(\mathbb{F}_q)/G(\varphi, +F_q)| = 1$$

and so we take $f(\mu, \varphi, 1, \lambda) = 1$ and (10.1) again holds.