MY FAVORITE NUMBERS:

John Baez

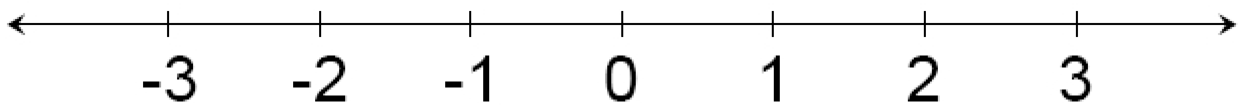
April 15, 2015

Sponsored by the University of York Maths Society

For more, visit http://tinyurl.com/baez-york-8

1: THE REAL NUMBERS

Once upon a time, numbers formed a line:



We call these the *real numbers*.

2: THE COMPLEX NUMBERS

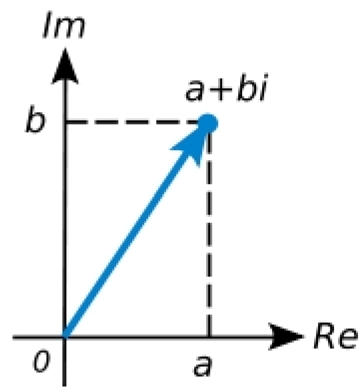
In 1545, Cardano published a book that showed how to solve cubic equations with the help of an imaginary number i with a shocking property:

$$i^2 = -1$$

'Complex numbers' like $a + bi$ gradually caught on, but people kept wondering:

Does the number i 'really exist'? If so, what is it?

In 1806, Argand realized you can draw complex numbers as points in the plane:



Multiplying by $a + bi$ simply amounts to rotating and expanding/shrinking the plane to make the number 1 go to the number $a + bi$.

So: you can *divide* by any nonzero complex number just by undoing the rotation and dilation it causes!

In 1835, William Rowan Hamilton realized you could treat complex numbers as pairs of real numbers:

$$\boldsymbol{a + bi = (a, b)}$$

Since real numbers work so well in 1d geometry, and complex numbers work so well in 2d geometry, Hamilton tried to invent 'triplets' for 3d geometry!

$$\boldsymbol{a + bi + cj = (a, b, c)}$$

His quest built to its climax in October 1843:

Every morning in the early part of the above-cited month, on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me: "Well, Papa, can you multiply triplets?" Whereto I was always obliged to reply, with a sad shake of the head: "No, I can only add and subtract them."

Hamilton seems to have wanted a 3-dimensional 'normed division algebra':

A *normed division algebra* is a finite-dimensional real vector space A with a *product* $A \times A \to A$, *identity* $1 \in A$ and *norm* $|\,|: A \to [0, \infty)$ such that:

$$1a = a = a1$$

$$a(b+c) = ab + ac \qquad (b+c)a = ba + ca$$

$$(\alpha a)b = \alpha(ab) = a(\alpha b) \qquad (\alpha \in \mathbb{R})$$

$$|a+b| \leq |a| + |b|$$

$$|a| = 0 \iff a = 0$$

and

$$|ab| = |a||b|$$

Hamilton didn't know it, but:

Theorem: **A normed division algebra must have dimension 1, 2, 4, or 8.**

4: THE QUATERNIONS

On October 16th, 1843, walking with his wife along the Royal Canal in Dublin, Hamilton suddenly found a 4-dimensional normed division algebra he later called the *quaternions*:

That is to say, I then and there felt the galvanic circuit of thought close; and the sparks which fell from it were the fundamental equations between i, j, k; exactly such as I have used them ever since:

$$i^2 = j^2 = k^2 = ijk = -1$$

And in a famous act of mathematical vandalism, he carved these equations into the stone of the Brougham Bridge:

Hamilton spent the rest of his life working on quaternions. They neatly combine scalars and vectors:

$$a = \underbrace{a_0}_{\text{scalar part}} + \underbrace{a_1 i + a_2 j + a_3 k}_{\text{vector part, } \vec{a}}$$

$$|a| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$$

Since

$$ij = k = -ji \qquad jk = i = -kj \qquad ki = j = -ik$$

$$i^2 = j^2 = k^2 = -1$$

we can show

$$ab = (a_0 b_0 - \vec{a} \cdot \vec{b}) + (a_0 \vec{b} + b_0 \vec{a} + \vec{a} \times \vec{b})$$

But the dot product and cross product of vectors were only isolated *later*, by Gibbs. Before 1901, quaternions reigned supreme!

8: THE OCTONIONS

The day after his fateful walk, Hamilton sent his college friend John T. Graves an 8-page letter describing the quaternions. On October 26th Graves replied, complimenting Hamilton on his boldness, but adding:

There is still something in the system which gravels me. I have not yet any clear views as to the extent to which we are at liberty arbitrarily to create imaginaries, and to endow them with supernatural properties.

If with your alchemy you can make three pounds of gold, why should you stop there?

On the day after Christmas, Graves sent Hamilton a letter about an 8-dimensional normed division algebra he called the *octaves* — now known as the *octonions*.

In January 1844 he sent Hamilton 3 more letters. He tried to construct a 16-dimensional normed division algebra, but "met with an unexpected hitch" and came to doubt this was possible.

The quaternions are *noncommutative:*

$$ab \neq ba$$

In June, Hamilton noted that the octonions are also *nonassociative:*

$$(ab)c \neq a(bc)$$

Hamilton offered to publicize Graves' work, but never got around to it. He was too distracted by work on the quaternions.

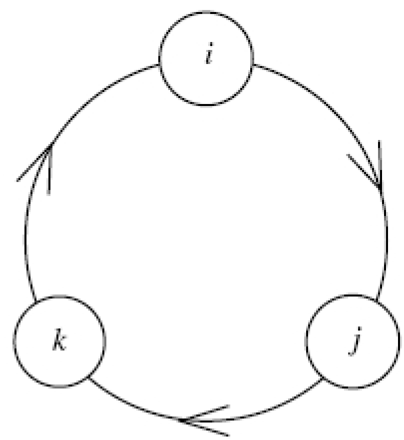
In 1845, the octonions were rediscovered by Arthur Cayley. So, some people call them *Cayley numbers*.

But, what *are* the octonions?
And why do they make the number 8 so special?

To multiply quaternions, you just need to remember:

- **1 is the multiplicative identity,**
- $\boldsymbol{i, j}$**, and** $\boldsymbol{k}$ **are square roots of -1,**

and this picture:

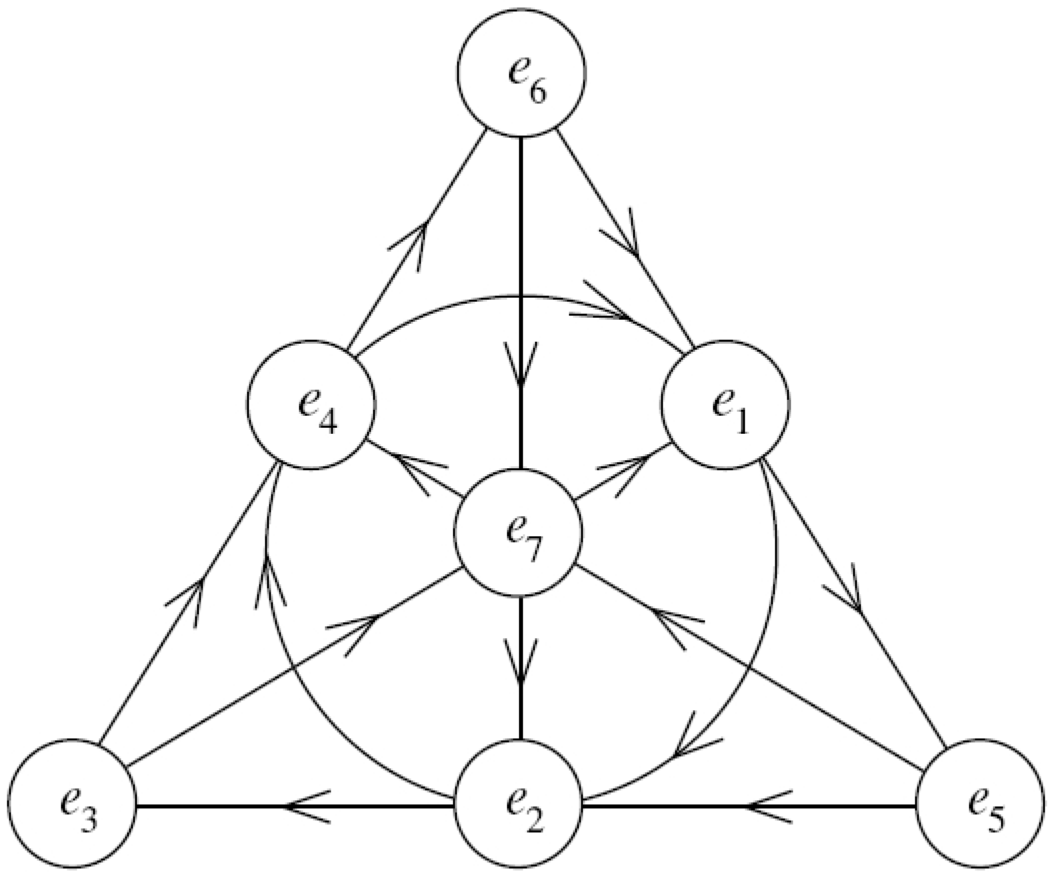


When we multiply two guys following the arrows we get the next one: for example, $\boldsymbol{jk = i}$**. But when we multiply going against the arrows we get** ***minus*** **the next one:** $\boldsymbol{kj = -i}$**.**

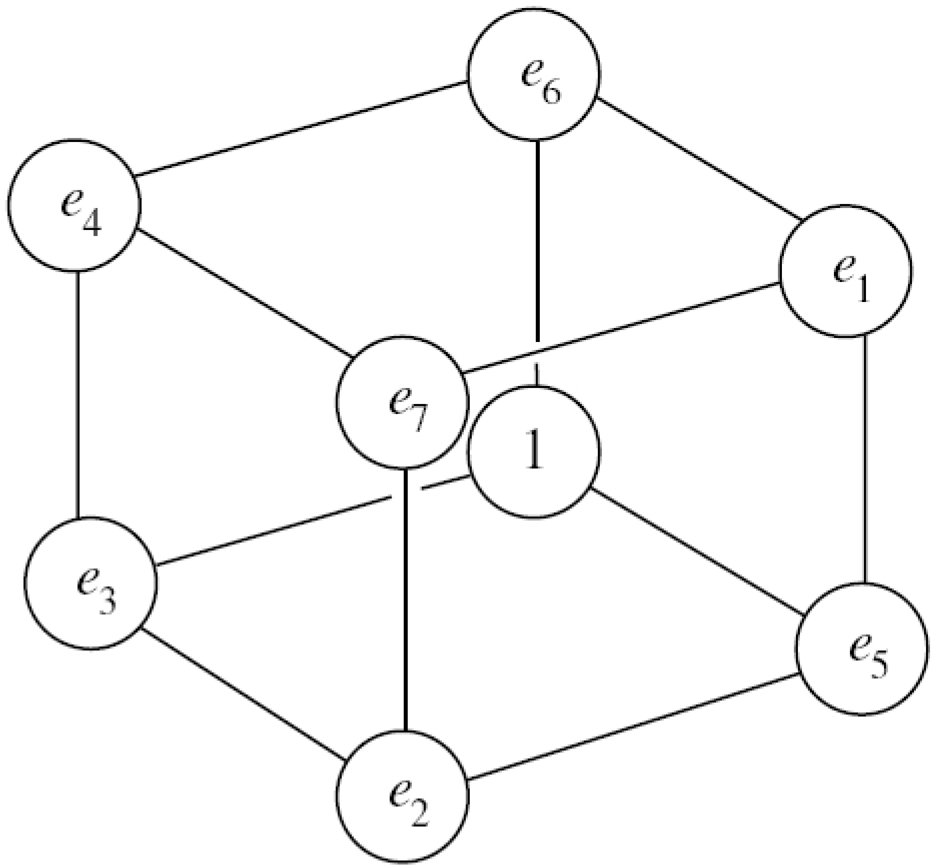
To multiply octonions, you just need to remember:

- **1 is the multiplicative identity**
- $e_1, \ldots, e_7$ **are square roots of -1**

and this picture of the *Fano plane:*



Points in the Fano plane correspond to lines through the origin in this cube:

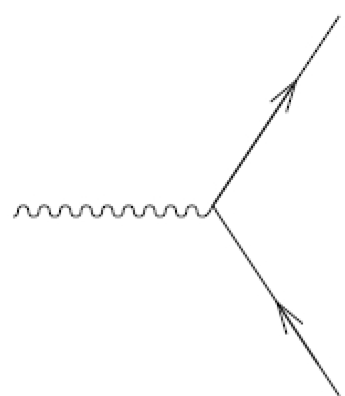


Lines in the Fano plane correspond to planes through the origin in this cube.

VECTORS VERSUS SPINORS

Force-carrying particles (like photons) are described by *vectors*, but articles of matter (like electrons) are described by *spinors*.

There's a way to 'multiply' a spinor and a vector and get a spinor:



When the space of spinors and the space of vectors have the same dimension, this multiplication gives a normed division algebra!

n	vectors	spinors	normed division algebra?
1	$\mathbb{R}$	$\mathbb{R}$	YES: REAL NUMBERS
2	$\mathbb{R}^2$	$\mathbb{R}^2$	YES: COMPLEX NUMBERS
3	$\mathbb{R}^3$	$\mathbb{R}^4$	NO
4	$\mathbb{R}^4$	$\mathbb{R}^4$	YES: QUATERNIONS
5	$\mathbb{R}^5$	$\mathbb{R}^4$	NO
6	$\mathbb{R}^6$	$\mathbb{R}^4$	NO
7	$\mathbb{R}^7$	$\mathbb{R}^8$	NO
8	$\mathbb{R}^8$	$\mathbb{R}^8$	YES: OCTONIONS

Bott periodicity: **spinors in dimension 8 more have dimension 16 times as big.**

So, we only get 4 normed division algebras.

SUPERSTRINGS

In string theory, different ways a string can vibrate correspond to different particles.

Strings trace out 2-dimensional surfaces in spacetime. So, a string in $(n+2)$-dimensional spacetime can vibrate in n directions perpendicular to this surface.

'Supersymmetric' strings are possible when the space of n-dimensional vectors has the same dimension as the space of spinors. There are only four options:

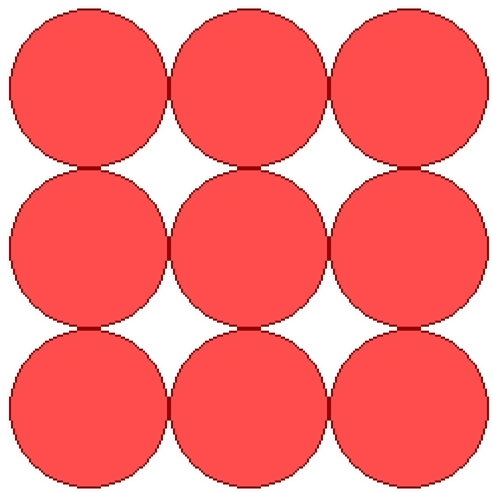
$$\begin{array}{lll} n = 1 \implies n+2 = 3 & \text{(real numbers)} \\ n = 2 \implies n+2 = 4 & \text{(complex numbers)} \\ n = 4 \implies n+2 = 6 & \text{(quaternions)} \\ n = 8 \implies n+2 = 10 & \text{(octonions)} \end{array}$$

**For reasons I don't fully understand,
only $n = 8$ gives a well-behaved superstring theory
when we take quantum mechanics into account.**

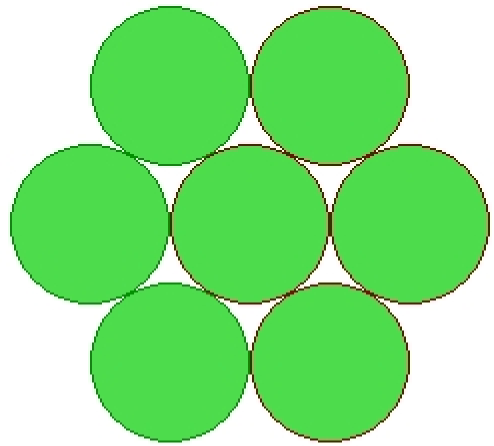
***So: if superstring theory is right,
spacetime is 10-dimensional,
and the vibrations of the strings
that make up all forces and particles
are described by octonions!***

SPHERE PACKING

The two most symmetrical ways to pack pennies are a square lattice, called $\mathbb{Z}^2$:

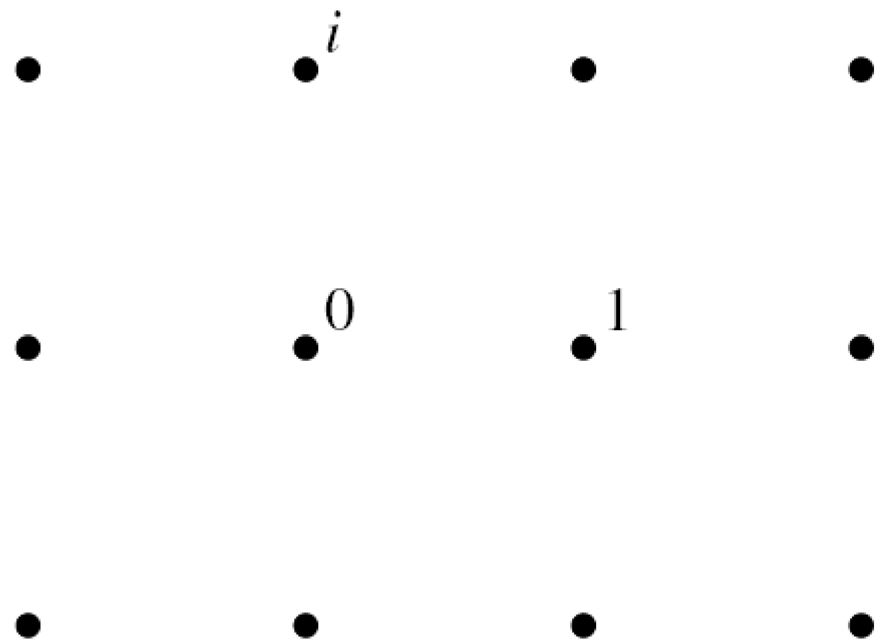


...and a hexagonal lattice, called A_2:



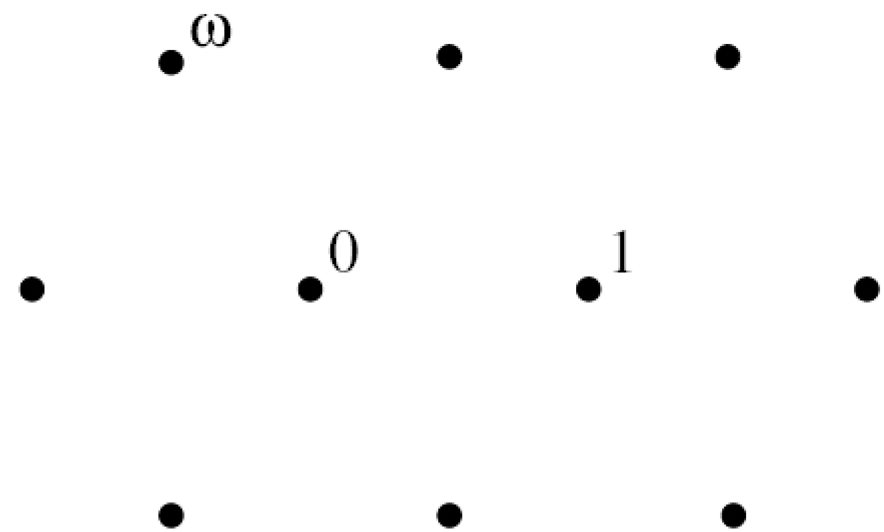
The A_2 packing is denser —
the densest possible in 2d.

The centers of pennies in the $\mathbb{Z}^2$ lattice are also special complex numbers called *Gaussian integers:*



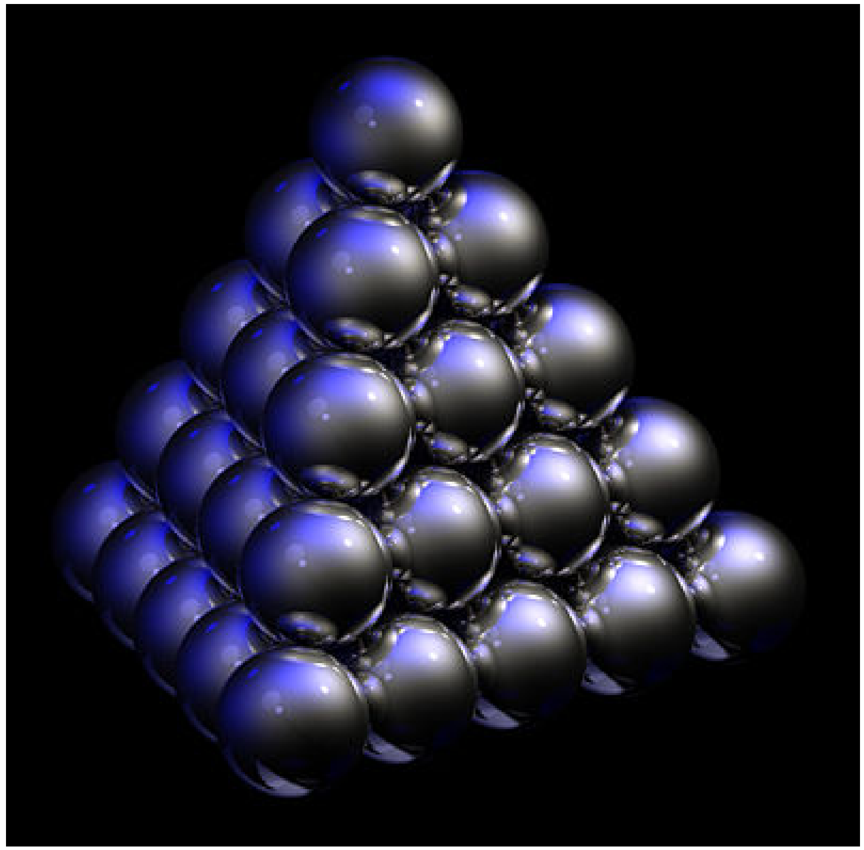
They're closed under addition and multiplication!

The centers of pennies in the A_2 lattice are special complex numbers called ***Eisenstein integers:***



They're ***also*** closed under addition and multiplication, since $\omega^3 = 1$.

In 3 dimensions, we can pack spheres in a cubical lattice called $\mathbb{Z}^3$, or a denser A_3 lattice made by stacking hexagonal lattices:



The A_3 lattice is the densest possible in 3d.

$\mathbb{Z}^n$ and A_n lattices exist in any dimension, but the densities drop:

n	$\mathbb{Z}^n$ density	A_n density
1	100%	100%
2	79%	91%
3	52%	74%
4	31%	55%

However, in 4d there's a surprise!

The spaces between spheres in the $\mathbb{Z}^4$ lattice are big enough to slip *another copy* of this lattice in the gaps, thus *doubling the density!*

We have already have spheres centered at points with integer coordinates:

$$(n_1,\ n_2,\ n_3,\ n_4)$$

Each center has distance 1 from its nearest neighbors. New spheres centered at points

$$(n_1 + 1/2,\ n_2 + 1/2,\ n_3 + 1/2,\ n_4 + 1/2)$$

will be *just as far away*, since

$$\sqrt{(1/2)^2 + (1/2)^2 + (1/2)^2 + (1/2)^2} = 1$$

We call this new lattice the D_4 lattice: it's the densest possible in 4d.

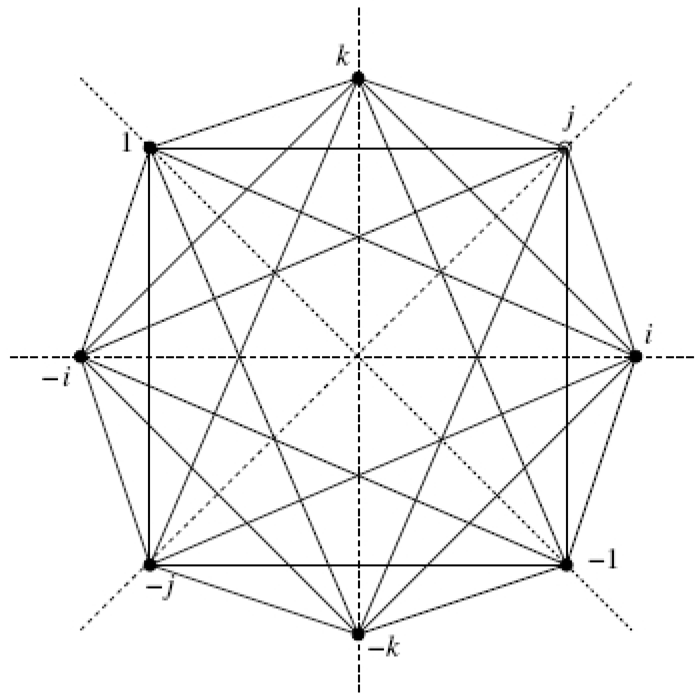
Points in the D_4 lattice are also special quaternions called *Hurwitz integers:*

$$a + bi + cj + dk$$

where a, b, c, d are either all integers or all integers plus 1/2. They're closed under addition and multiplication!

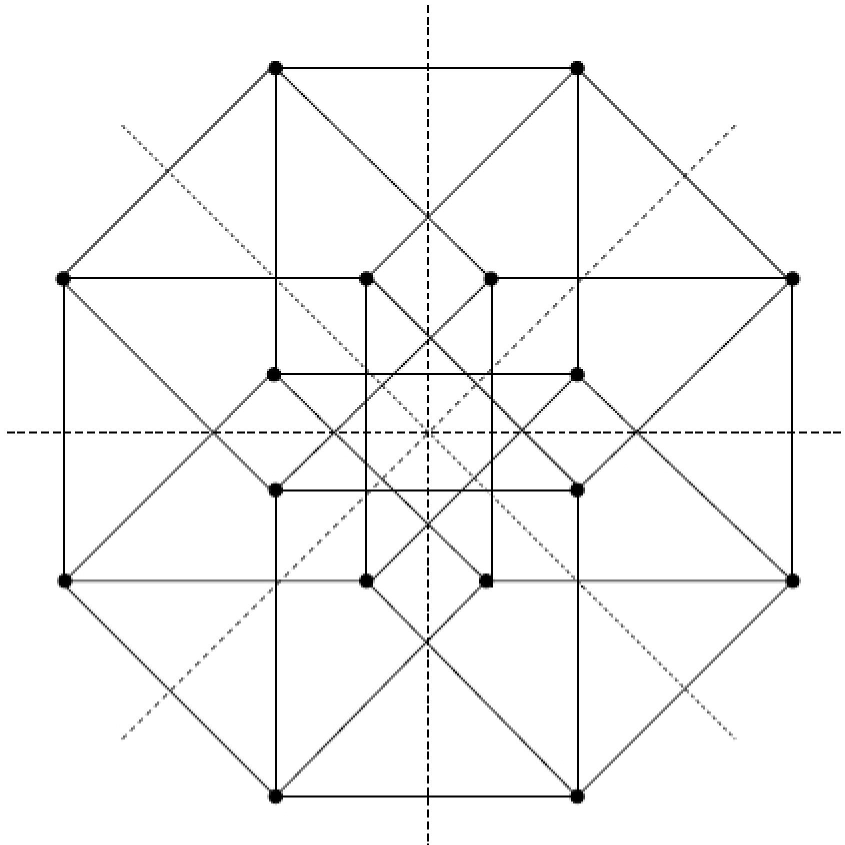
Each Hurwitz integer has 24 nearest neighbors, so each sphere in the D_4 lattice touches 24 others. The 24 nearest neighbors of 0 are...

8 like this...



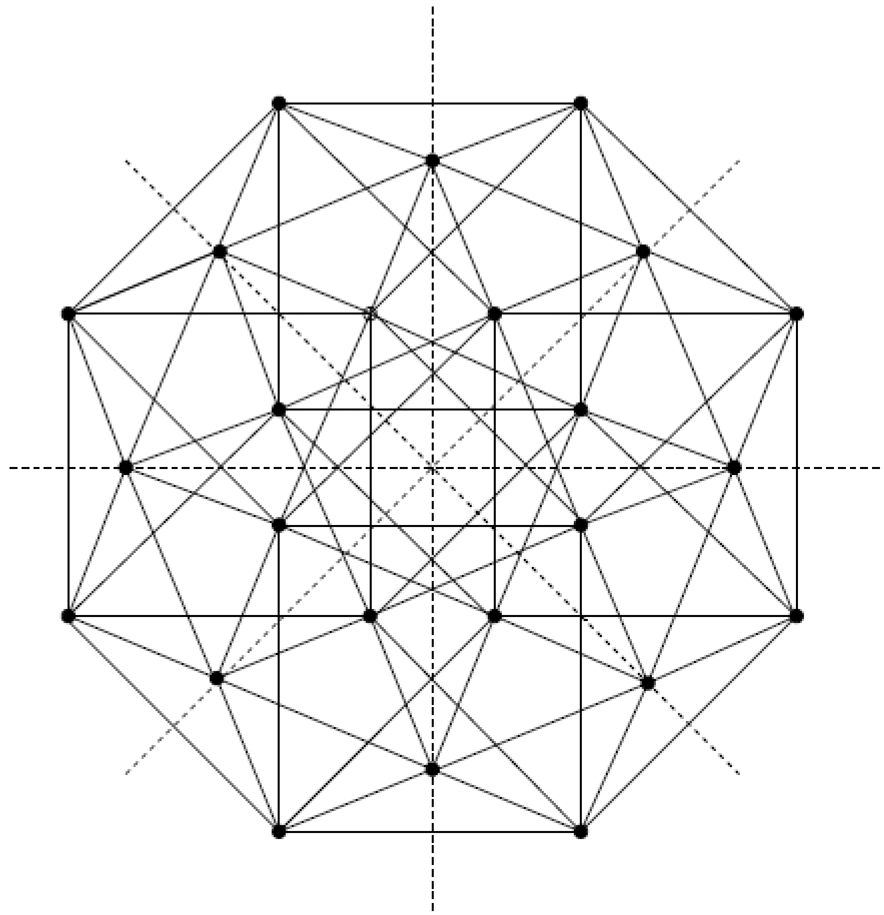
$$\pm 1,\ \pm i,\ \pm j,\ \pm k$$

and 16 like this...



$$\pm\frac{\mathbf{1}}{\mathbf{2}}\pm\frac{\boldsymbol{i}}{\mathbf{2}}\pm\frac{\boldsymbol{j}}{\mathbf{2}}\pm\frac{\boldsymbol{k}}{\mathbf{2}}$$

...for a total of 24:



D_n lattices exist in any dimension ≥ 4, but the densities drop:

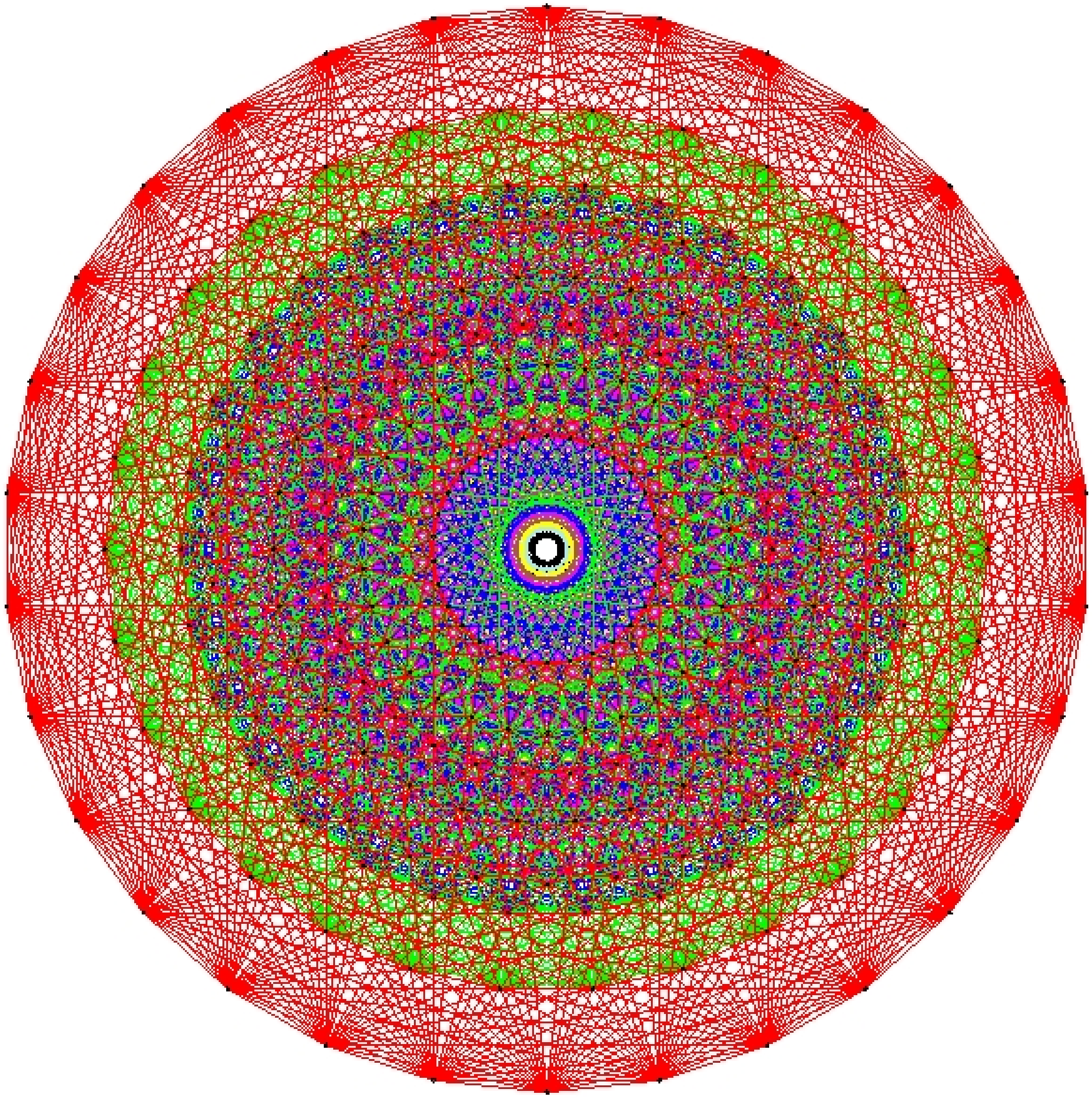
n	$\mathbb{Z}^n$ density	A_n density	D_n density
1	100%	100%	
2	79%	91%	
3	52%	74%	
4	31%	55%	62%
5	16%	38%	47%
6	8%	24%	32%
7	4%	15%	21%
8	2%	8%	13%

However, in 8 dimensions there's another surprise!

**There's enough space between spheres in the D_8 lattice to slip *another copy* in the gaps!
We get a lattice called the E_8 lattice.
It's the densest possible in 8 dimensions.**

**Points in the E_8 lattice are also special octonions called *Cayley integers*.
They're closed under addition and multiplication!**

Each Cayley integer has 240 nearest neighbors, so each sphere in the E_8 lattice touches 240 others.



The E_8 lattice also seems to be important in string theory!

But this is the beginning of another,
longer,
stranger,
still unfinished
story...

For more, visit http://tinyurl.com/baez-york-8

APPENDIX: LATTICES

The densest lattices in ≤ 8 dimensions are among these:

n	$\mathbb{Z}^n$ density	A_n density	D_n density	E_n density
1	100%	100%		
2	79%	91%		
3	52%	74%		
4	31%	55%	62%	
5	16%	38%	47%	
6	8%	24%	32%	37%
7	4%	15%	21%	30%
8	2%	8%	13%	25%

The lattices E_6 and E_7 are constructed as lower-dimensional 'slices' of E_8.

APPENDIX: QUATERNIONS, THE DODECAHEDRON, AND E_8

We call a quaternion q with $|q| = 1$ a *unit quaternion*. These form a group under multiplication.

Any unit quaternion q gives a rotation. As we've seen, a quaternion whose scalar part is zero is the same as a vector:

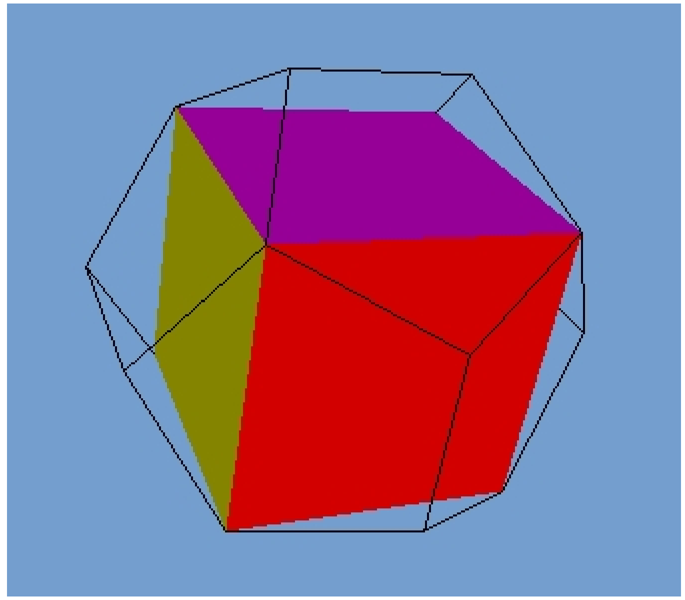
$$a_1 i + a_2 j + a_3 k = \vec{a}$$

To rotate this vector, we just 'conjugate' it by q:

$$q\vec{a}q^{-1}$$

We get all rotations this way. Both q and $-q$ give the same rotation, so the unit quaternions form the 'double cover' of the 3d rotation group. This double cover is usually called SU(2).

There are 5 ways to inscribe a cube in the dodecahedron:



Rotational symmetries of the dodecahedron give all *even* permutations of these 5 cubes, so these symmetries form what is called the 'alternating group' A_5, with

$$5!/2 = 60$$

elements.

Since the group SU(2) is the double cover of the 3d rotation group, there are

$$2 \times 60 = 120$$

unit quaternions that give rotational symmetries of the dodecahedron. These form a group usually called the *binary icosahedral group*, since the regular icosahedron has the same symmetries as the regular dodecahedron.

A wonderful fact: elements of the binary icosahedral group are precisely the unit quaternions

$$q = q_0 1 + q_1 i + q_2 j + q_3 k$$

where q_0, q_1, q_2, q_3 lie in the 'golden field'. The *golden field* consists of real numbers

$$x + \sqrt{5} y$$

where x and y are rational.

This gives another way to construct the E_8 lattice. A finite linear combination of the 120 unit quaternions described above is called an *icosian*. It takes 8 rational numbers to describe an icosian. But, not every 8-tuple of rational numbers gives an icosian. Those which do form a copy of the E_8 lattice!

To get this to work, we need to put the right norm on the icosians. First there is usual quaternionic norm, with

$$|a_0 + a_1 i + a_2 j + a_3 k|^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2$$

But for an icosian, this is always of the form $x + \sqrt{5}y$ for some rational x and y. We can define another norm on the icosians by setting

$$|a_0 + a_1 i + a_2 j + a_3 k|^2 = x + y$$

With this norm, the icosians form a copy of E_8 lattice.

There's also another relation between the dodecahedron and E_8, called the ***McKay correspondence***. This requires some more advanced math to understand, so I'll quickly sketch it and then quit.

The binary icosahedral group has 9 irreducible representations, one of them being the obvious representation on $\mathbb{C}^2$ coming from the 2d rep of SU(2). Draw a dot for each rep R_i; tensor each R_i with the rep on $\mathbb{C}^2$ and write the result as a direct sum of reps R_j; then draw a line connecting the dot for R_i to the dot for each R_j that shows up in this sum.

You get this picture:

Voilà! This is the extended E_8 Dynkin diagram! From this, we can recover E_8.

CREDITS AND NOTES

1. SoftCurve 4-inch nickel number 8, available from Home Depot, http://www.polyvore.com/cgi/thing?id=968851.
2. On-line Math Worksheet Generator: number line, The Math Worksheet Site, http://themathworksheetsite.com/numline.html
3. Text, John Baez.
4. Illustration of a complex number created by 'Wolfkeeper', from the Wikipedia article Complex Number.
5. Text, John Baez.
6. Quote of Hamilton from Robert Perceval Graves, *Life of Sir William Rowan Hamilton*, 3 volumes, Arno Press, New York, 1975.
7. Text, John Baez.
8. The classification of normed division algebras was first proved by Adolf Hurwitz, in Über die Composition der quadratischen Formen von beliebig vielen Variabeln, *Nachr. Ges. Wiss. Göttingen* (1898), 309–316.

9. Quote of Hamilton from Robert P. Graves, *Life of Sir William Rowan Hamilton*, 3 volumes, Arno Press, New York, 1975.

10. Photograph of Brougham Brige by Tevian Dray, http://math.ucr.edu/home/baez/octonions/node24.html

11. Information about Gibbs from Michael J. Crowe, *A History of Vector Analysis*, University of Notre Dame Press, Notre Dame, 1967.

12. Quote of Hamilton from Robert P. Graves, *Life of Sir William Rowan Hamilton*, 3 volumes, Arno Press, New York, 1975.

13. Chronology from *Life of Sir William Rowan Hamilton*, *op. cit.*

14. Arthur Cayley, On Jacobi's elliptic functions, in reply to the Rev. B. Bronwin; and on quaternions, *Philos. Mag.* 26 (1845), 208–211.

    This article was so full of mistakes that only the appendix about octonions is reprinted in Cayley's collected works. Arthur Cayley, *The Collected Mathematical Papers*, Johnson Reprint Co., New York, 1963, p. 127.

15. Drawing from John Baez, The octonions, ***Bull. Amer. Math. Soc.*** 39 (2002), 145–205. Errata in ***Bull. Amer. Math. Soc.*** 42 (2005), 213. Also available at http://math.ucr.edu/home/baez/octonions/

16. Drawing from The octonions, ***op. cit.***

17. Drawing from The octonions, ***op. cit.***

18. Material from The octonions, ***op. cit.***

19. Material from The octonions, ***op. cit.***

20. The relation between supersymmetry and the division algebras is discussed in T. Kugo and P.–K. Townsend, Supersymmetry and the division algebras, ***Nucl. Phys.*** B221 (1983), 357–380. The existence of classical superstring Lagrangians only in spacetime dimensions 3, 4, 6 and 10 is discussed in M. B. Green, J. H. Schwarz and E. Witten, ***Superstring Theory***, vol. 1, Cambridge U. Press, 1987. See especially section 5.1.2, on the supersymmetric string action.

21. Text, John Baez.

22. Picture by John Baez.

23. Picture by John Baez. Proof that A_2 is the densest sphere packing in 2d by László Fejes Tóth, Über einen geometrischen Satz, *Math. Z.* 46 (1940), 79–83.

24. Picture from John Baez, *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry*, by John H. Conway and Derek A. Smith, review in *Bull. Amer. Math. Soc.* 42 (2005), 229–243. Also available at http://math.ucr.edu/home/baez/octonions/conway_smith/

25. Picture from review of *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry*, *op. cit.*

26. Picture of closed-packed spheres by 'Greg L.', from the Wikipedia article Close-packing.

    Proof that A_3 is the densest packing of spheres in 3d by Thomas Hales, available at http://www.math.pitt.edu/~thales/kepler98/

27. Table of sphere packing densities based on data from John H. Conway and Neil J. A. Sloane, *Sphere Packings, Lattices and Groups*, Springer, Berlin, 1998.

28. Text, John Baez.

29. Proof that D_4 is the densest possible *lattice* packing of spheres in 4d by Korkine and Zolotareff, Sur les formes quadratique positive quaternaires, *Math. Ann.* 5 (1872), 581–583. It is not known if there are denser non-lattice packings.
30. Drawing from review of *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry*, *op. cit.*
31. Drawing from review of *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry*, *op. cit.*
32. Drawing from review of *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry*, *op. cit.*
33. Table of sphere packing densities based on data from *Sphere Packings, Lattices and Groups*, *op. cit.*
34. Text, John Baez.
35. Drawing of E_8 in Coxeter plane by John Stembridge, available at http://www.math.lsa.umich.edu/~jrs/coxplane.html
36. Text, John Baez.

37. Table of sphere packing densities based on data from *Sphere Packings, Lattices and Groups*, *op. cit.*

38. Text, John Baez.

39. Joel Roberts, JGV example: cube in a dodecahedron, `http://www.math.umn.edu/~roberts/java.dir/JGV/cube-in-dodeca.html`

40. Discussion of the icosians and E_8 from *Sphere Packings, Lattices and Groups*, *op. cit.*

41. For a gentle introduction to the McKay correspondence see Joris van Hoboken, *Platonic Solids, Binary Polyhedral groups, Kleinian Singularities and Lie Algebras of Type A,D,E*, Master's Thesis, University of Amsterdam, 2002, available at `http://math.ucr.edu/home/baez/joris_van_hoboken_platonic.pdf`

For a more advanced overview also try John McKay, a rapid introduction to ADE theory, `http://math.ucr.edu/home/baez/ADE.html`

42. Extended E_8 Dynkin diagram, source unknown.