SPT order and algebraic topology

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Two kinds of quantum theories: H-type and L-type

- H-type:

- Hilbert space with tensor structure $\mathcal{V} = \otimes_i \mathcal{V}_i$
- A local Hamiltonian operator acting on $\mathcal{V}$
  $\to$ Space-time path integral well defined on mapping torus $M^{d-1}_{\text{space}} \rtimes S^1$.

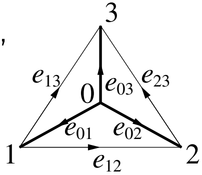
- L-type:

- Tensors for each cell in space-time lattice defined path integral
  $\to$ Space-time path integral well defined on any space-time manifold $M^d_{\text{space-time}}$.

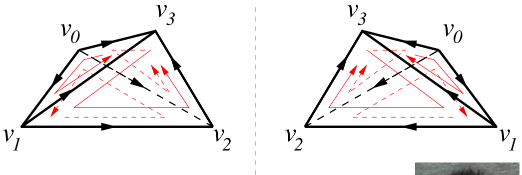
- Lead to different classification of topo. orders and SPT orders
  H-type models $\to$ E_8 bosonic quantum Hall state
  L-type models $\to$ $(E_8)^3$ bosonic quantum Hall state

L-type local bosonic quantum systems

- A L-type local **quantum system** in d-dimensional space-time M^d is described by (use $d = 3$ as an exmaple):



- a trianglation of M^3 with a branching structure,
- a set of indices $\{v_i\}$ on vertecies, $\{e_{ij}\}$ on edges, $\{\phi_{ijk}\}$ on triangles.
- two real and one complex tensors $T_3 = \{W_{v_0}, A^{e_{01}}_{v_0 v_1}(SO_{ij}), C_{\pm}{}^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23}}_{v_0 v_1 v_2 v_3;\phi_{123}\phi_{013}\phi_{023}\phi_{012}}(SO_{ij})\}$ for each $0, 1, 3$-cell (vertex, edge, tetrahedron).

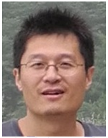


- **Unitary condition**

$$W_{v_0} > 0, \qquad A^{e_{01}}_{v_0 v_1} > 0,$$

$$C_{+}{}^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23}}_{v_0 v_1 v_2 v_3;\phi_{123}\phi_{013}\phi_{023}\phi_{012}} = \left(C_{-}{}^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23}}_{v_0 v_1 v_2 v_3;\phi_{123}\phi_{013}\phi_{023}\phi_{012}}\right)^*,$$

[Kong-Wen 14]

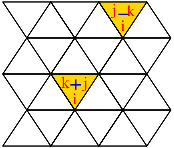


Partition function and Correlation function

- **Partition function**:

$$Z = \sum_{v_0,\cdots;e_{01},\cdots;\phi_{012},\cdots} \prod_{\text{vertex}} W_{v_0} \prod_{\text{edge}} A^{e_{01}}_{v_0 v_1} \prod_{\text{tetra}} C_{s_{0123}\, v_0 v_1 v_2 v_3;\phi_{123}\phi_{013}\phi_{023}\phi_{012}}^{e_{01}e_{02}e_{03}e_{12}e_{13}e_{23}}$$

$$= \sum \prod T_3$$



where $s_{0123} = \pm$ depends on the orientation of the tetrahedron.

- **Correlation function** on closed space-time manifold – physically measurible quantities:
  Modify tensor on a few simplices gives us a new partition function $T_3 \to \tilde{T}_3$: $Z(M^d) \to Z[\tilde{T}_3(x), \tilde{T}_3(y), \cdots; M^d]$:

$$\langle \tilde{T}_3(x)\tilde{T}_3(y)\rangle = \frac{Z[\tilde{T}_3(x), \tilde{T}_3(y); M^d]}{Z(M^d)}$$

Short-range correlated (SRC) system and liquid

- A **infinite-system** is not a single system but a sequence of systems

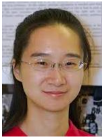
- with size of space-time $M^3 \to \infty$ and size of simplices ~ 1.
- each vertex is shared by at most a finite number of simplices.

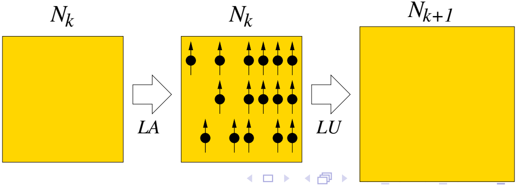
- A **short-range correlated** (SRC) infinite-system satisfies

(1) $$\langle \tilde{T}_3(x)\tilde{T}_3(y)\rangle - \langle \tilde{T}_3(x)\rangle\langle \tilde{T}_3(y)\rangle \sim \mathrm{e}^{-\frac{|x-y|}{\xi}}$$

for a fixed ξ in the infinite system-size limit;
(2) systems of different sizes in the sequence can deform into each other and keep the SRC property during the deformation.

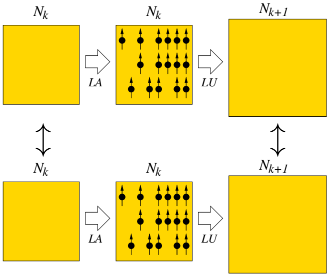
[Zeng-Wen 14] $\to$ **SRC liquid**





Short-range correlated (SRC) liquid phases

- An equivalence relation:
  Two SRC infinite-systems are equivalent if the two sequences can deform into each other while keeping the SRC property during the deformation.
- The resulting equivalent classes are **SRC liquid phases** or **L-type topological orders**
  [Wen 89]
  [Chen-Gu-Wen 10]



How to classify SRC liquid phases (L-type topological orders) in each dimension?

L-type local bosonic systems with symmetry

(1) The indices admit a symmetry action $v_i \to g \cdot v_i$, $e_{ij} \to g \cdot e_{ij}$, $\phi_{ijk} \to g \cdot \phi_{ijk}$, where $g \in G$.
(2) $T_{G,3} = \{W_{v_0}, A^{e_{01}}_{v_0 v_1}(SO_{ij}), C_{\pm}{}^{e_{01} e_{02} e_{03} e_{12} e_{13} e_{23}}_{v_0 v_1 v_2 v_3; \phi_{123} \phi_{013} \phi_{023} \phi_{012}}(SO_{ij})\}$ are invariant under the above symmetry action.

- SRC liquid phases with symmetry:
  Equivalence relation: Two symmetric SRC infinite-systems can deform into each other while keeping the symmetry property and the SRC property during the deformation.
  How to classifies SRC liquid phases with symmetry $\to$ L-type symmetry enriched topological (SET) order

Examples of SRC liquids

- **Example 1:** Tensors: non-zero only when all the indices are 1
  $W_1 = w_0, \; A^1_{11} = w_1, \; C(\{v_i = 1, e_{ij} = 1, \phi_{ijk} = 1\}) = w_3,$
  All degrees of freedom are frozen to 1.
  $\rightarrow \qquad Z(M^3) = \prod_{n=0}(w_n)^{N_n}$

- The above systems with different w_n's all belong to the same SRC phase [have a trivial topological order (triTO)].

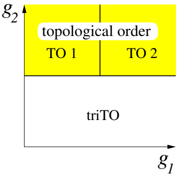
- **Example 2:** Tensors: non-zero only when all the edge-indices, face indices, *etc* are 1; the vertex-indices $v_i \in G$
  $W_{v_i} = w_0, \; A^1_{v_i v_j} = w_1 \; C(\{v_i, e_{ij} = 1, \phi_{ijk} = 1\}) = w_3,$
  $\rightarrow \qquad Z(M^3) = |G|^{N_0} \prod_{n=0}(w_n)^{N_n}$

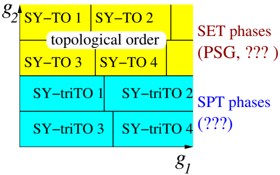
- The above systems have a symmetry G

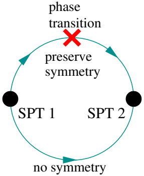
- A **conjecture**: A system with $Z(M^d) = 1$ for all closed orientable space-time M^d has a trivial topological order.
  Both of the above examples have a trivial topological order.

A general picture for SRC phases

- **Non trivial TO w/o symm.** → many phases [Wen 89]
  Trivial TO w/o symm. → one phase (no symm. breaking)
- **Non trivial TO with symm.** → many phases [Wen 02]
  Trivial TO with symm. → many different phases [Gu-Wen 09]
  may be called **symmetry protected trivial (SPT)** phase
  or **symmetry protected topological (SPT)** phase





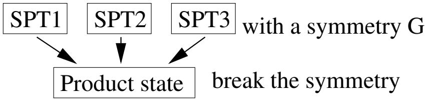


- SPT phases = equivalent class of *symm.* smooth deformation
- Examples: 1D Haldane phase[Haldane 83] 2D/3D TI[Kane-Mele 05; Bernevig-Zhang 06] [Moore-Balents 07; Fu-Kane-Mele 07]

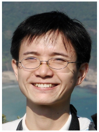
1D  2D   3D  

SPT phases – the trivial TO with symmetry

SPT phases are SRC (gapped) quantum phases with a certain symmetry, which can be smoothly connected to the same trivial phase if we remove the symmetry $\to$ trivial topo order.







- **A group cohomology thepry of SPT phases**:
  Using each element in $\mathcal{H}^d[G, U(1)]$ (the d group cohomology class of the group G with $U(1)$ as coefficient), we can construct a exactly soluble path integral in d-dimensional space-time, which realize a SPT state with a symmetry G. [Chen-Gu-Liu-Wen 11]

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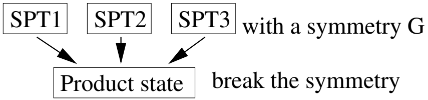

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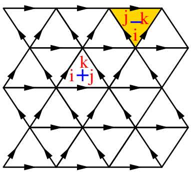
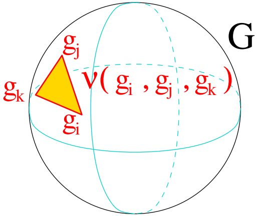
  [Chen-Gu-Liu-Wen 11]
- How to get the above result? Construct a path integral with symmetry G and $Z_{\text{top}}(M^d) = 1$ on any closed orientable M^d.

L-type SPT state on 1+1D space-time lattice

- Path integral on 1+1D space-time lattice described by tensors
  $T_2 = \{W_{v_0}, B_{\pm v_0 v_1 v_2}^{e_{01} e_{12} e_{02}}(SO_{ij})\}_{e_{ij}=1} = \{W_{v_0} = |G|^{-1}, [\nu_2(v_0, v_1, v_2)]^{1,*}\}$
  $\mathrm{e}^{-S} = \prod \nu_2^{s_{ijk}}(g_i, g_j, g_k),\ Z = |G|^{-N_0} \sum \mathrm{e}^{-S},\ v_i \to g_i,\ g_i \in G$
  where $\nu^{s_{ijk}}(g_i, g_j, g_k) = e^{-\int_\triangle L}$ and $s_{ijk} = 1, *$
- The above defines a LNσM with target space G on 1+1D space-time lattice.



space–time

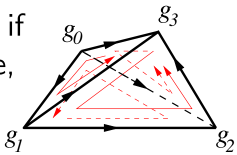


- The NLσM will have a symmetry G if $g_i \in G$ and

$$\nu_2(g_i, g_j, g_k) = \nu_2(hg_i, hg_j, hg_k), \quad h \in G$$

- We will get a SPT state if we choose $\nu_2(g_i, g_j, g_h) = 1 \to$ $Z_{\text{top}}(M^2) = 1$ (which is a trivial SPT state).

Topological path inegral (NLσM) and SPT state



- $\nu(g_i, g_j, g_k)$ give rise to a topological NLσM if $\mathrm{e}^{-S_{\text{fixed}}} = \prod \nu^{s_{ijk}}(g_i, g_j, g_k) = 1$ on any sphere, including a tetrahedron (simplest sphere).
- $\nu(g_i, g_j, g_k) \in U(1)$
- On a tetrahedron $\to$ 2-cocycle condition

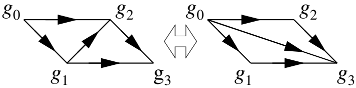
$$\nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)\nu_2^{-1}(g_0, g_2, g_3)\nu_2^{-1}(g_0, g_1, g_2) = 1$$

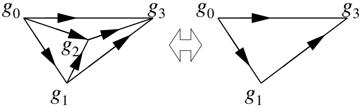
  The solutions of the above equation are called **group cocycle**.
- The 2-cocycle condition has many solutions: $\nu_2(g_0, g_1, g_2)$ and $\tilde{\nu}_2(g_0, g_1, g_2) = \nu_2(g_0, g_1, g_2)\frac{\beta_1(g_1,g_2)\beta_1(g_0,g_1)}{\beta_1(g_0,g_2)}$ are both cocycles. We say $\nu_2 \sim \tilde{\nu}_2$ (equivalent).
- The set of the equivalent classes of ν_2 is denoted as

$$\mathcal{H}^2[G, U(1)] = \pi_0(\text{space of the solutions}).$$

- $\mathcal{H}^2[G, U(1)]$ describes 1+1D SPT phases protected by G.

Topological invariance in topological NLσMs





As we change the space-time lattice,
the action amplitude e^{-S} does not change:

$$\nu_2(g_0,g_1,g_2)\nu_2^{-1}(g_1,g_2,g_3) = \nu_2(g_0,g_1,g_3)\nu_2^{-1}(g_0,g_2,g_3)$$

$$\nu_2(g_0,g_1,g_2)\nu_2^{-1}(g_1,g_2,g_3)\nu_2(g_0,g_2,g_3) = \nu_2(g_0,g_1,g_3)$$

as implied by the cocycle condition:

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The topological NLσM is a RG fixed-point.

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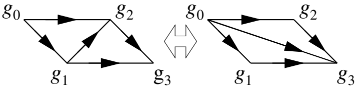

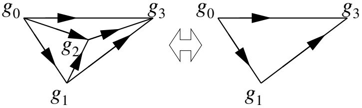

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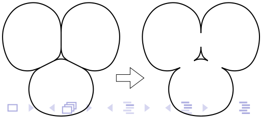
$$\nu_2(g_1,g_2,g_3)\nu_2(g_0,g_1,g_3)\nu_2^{-1}(g_0,g_2,g_3)\nu_2^{-1}(g_0,g_1,g_2) = 1$$

The topological NLσM is a RG fixed-point.

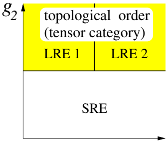
- The $Z(M^d) = 1$ if M^d can be obtained by gluing spheres. The NLσM describes a SPT state with trivial topo. order.

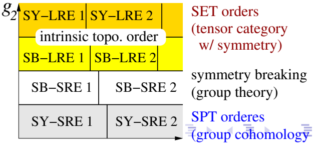


Bosonic SPT phases from $\mathcal{H}^d[G, U(1)]$

Symmetry G / $d=$	$0+1$	$1+1$	$2+1$	$3+1$
$U(1) \rtimes Z_2^T$ (top. ins.)	$\mathbb{Z}$	$\mathbb{Z}_2$ (0)	$\mathbb{Z}_2$ $(\mathbb{Z}_2)$	$\mathbb{Z}_2^2$ $(\mathbb{Z}_2)$
$U(1) \rtimes Z_2^T \times$ trans	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}_2^8$
$U(1) \times Z_2^T$ (spin sys.)	0	$\mathbb{Z}_2^2$	0	$\mathbb{Z}_2^3$
$U(1) \times Z_2^T \times$ trans	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^9$
Z_2^T (top. SC)	0	$\mathbb{Z}_2$ $(\mathbb{Z})$	0 (0)	$\mathbb{Z}_2$ (0)
$Z_2^T \times$ trans	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$
$U(1)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$U(1) \times$ trans	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^4$
Z_n	$\mathbb{Z}_n$	0	$\mathbb{Z}_n$	0
$Z_n \times$ trans	$\mathbb{Z}_n$	$\mathbb{Z}_n$	$\mathbb{Z}_n^2$	$\mathbb{Z}_n^4$
$D_{2h} = Z_2 \times Z_2 \times Z_2^T$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$
$SO(3)$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
$SO(3) \times Z_2^T$	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$

"Z_2^T": time reversal,
"trans": translation,
$0 \to$ only trivial phase.
$(\mathbb{Z}_2) \to$ free fermion result





Universal probe for SPT orders

- How do you know the constructed NLσM ground states carry non-trivial SPT order? How do you probe/measure SPT order?

  Universal probe = one probe to detect all possible orders.

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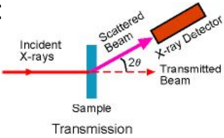
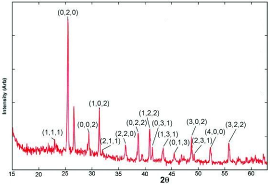
- Universal probe for crystal order = X-ray diffraction:





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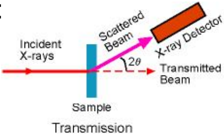

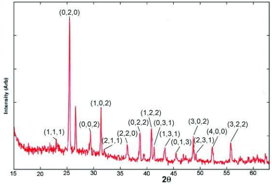

- Partitional function as an universal probe,
  but $Z_{\text{top}}^{SPT}(M^d) = 1 \to$ does not work.
- Twist the symmetry by "gauging" the symmetry on M^d
  $\to A - G$ gauge field.
  $\to Z_{\text{top}}^{SPT}(A, M^d) \neq 1$.
  [Levin-Gu 12; Hung-Wen 13]

Universal topo. inv.: “gauged” partition function

$$\frac{Z(A,M^d)}{Z(0,M^d)} = \frac{\int Dg\, \mathrm{e}^{-\int \mathcal{L}(g^{-1}(\mathrm{d}-\mathrm{i}A)g)}}{\int Dg\, \mathrm{e}^{-\int \mathcal{L}(g^{-1}\mathrm{d}g)}} = \mathrm{e}^{-\mathrm{i}2\pi \int W_{\text{topinv}}(A)}$$

- $W_{\text{topinv}}(A)$ and $W'_{\text{topinv}}(A)$ are equivalent if

$$W'_{\text{topinv}}(A) - W_{\text{topinv}}(A) = \frac{1}{\lambda_g}\mathrm{Tr}(F^2) + \cdots$$

- The equivalent class of the gauge-topological term $W_{\text{topinv}}(A)$ is the topological invariant that probe different SPT state.
- The topological invariant $W_{\text{topinv}}(A)$ are Chern-Simons terms or Chern-Simons-like terms.
- Such Chern-Simons-like terms are classified by
  $H^{d+1}(BG,\mathbb{Z}) = \mathcal{H}^d[G,U(1)]$ [Dijkgraaf-Witten 92]
  The topological invariant $W_{topinv}(A)$ can probe all the SPT states (constructed so far)

$U(1)$ SPT phases and their physical properties

- Topo. terms for $U(1)$ SPT state: $a \equiv \frac{A}{2\pi}$, $c_1 \equiv \frac{\mathrm{d}A}{2\pi}$, $ac_1 \equiv a \wedge c_1$

$d =$	$\mathcal{H}^d[U(1)]$	W^d_{topinv}
$0+1$	$\mathbb{Z}$	a
$1+1$	0	
$2+1$	$\mathbb{Z}$	ac_1
$3+1$	0	

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- In $0+1$D, $W^1_{\text{topinv}} = k\frac{A}{2\pi} = ka$.

$d =$	$\mathcal{H}^d[U(1)]$	W^d_{topinv}
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$1+1$	0	
$2+1$	$\mathbb{Z}$	ac_1
$3+1$	0	

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  $\to$ ground state carries charge k

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  $\to$ The edge of $U(1)$ SPT phase must be gapless: left/right movers + anomalous $U(1)$ symm.

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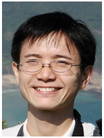
- **Probe:** $2\pi m$ flux in space M^2 induces km unit of charge $\to$ Hall conductance $\sigma_{xy} = 2ke^2/h$.
- **Mechanism:** Start with 2+1D bosonic superfluid:
  proliferate vortices $\to$ trivial Mott insulator.
  proliferate vortex+$2k$-charge $\to$ $U(1)$ SPT state labeled by k.

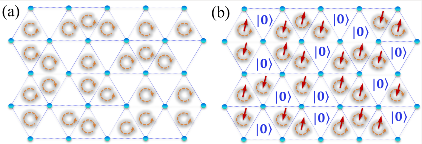
A mechanism for 2+1D $U(1) \rtimes Z_2^T$ SPT state



- 2+1D boson superfluid + gas of vortex
  $\to$ boson Mott insulator.
- 2+1D boson superfluid + gas of S^z-vortex
  $\to$ boson topological insulator ($U(1) \rtimes Z_2^T$ SPT state)

\- The boson superfluid + spin-1 system

S^z-vortex =
vortex + $(S_z = +1)$-spin
anti S^z-vortex =
anti-vortex + $(S_z = -1)$-spin



Probing 2+1D $U(1) \rtimes Z_2^T$ SPT state [Liu-Gu-Wen 14]

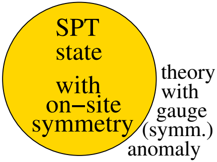
Let Φ_{vortex} be the creation operator of the vortex. Then

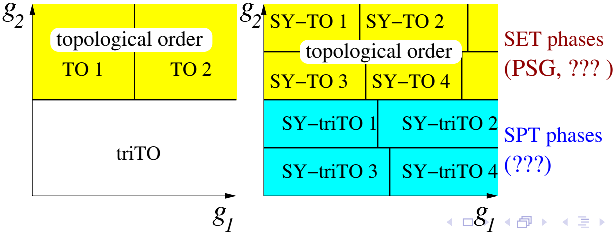
$$T^{-1}\Phi_{\text{vortex}}T = \Phi^\dagger_{\text{vortex}}, \text{ or } T^{-1}\Phi_{S_z\text{-vortex}}T = -\Phi^\dagger_{S_z\text{-vortex}}.$$

$U(1)$ π-flux in the non-trivial SPT phase is a Kramer doublet

$\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ does not produce all the SPT phases with symm. G: Topological states and anomalies

SPT order from $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$





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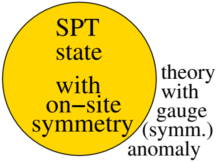
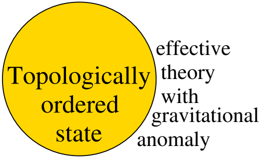
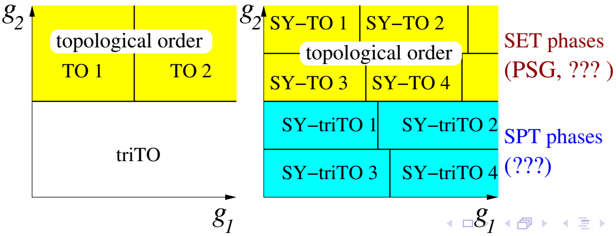


Topological order





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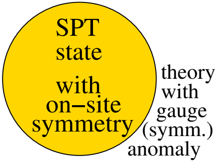
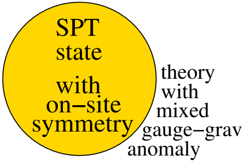
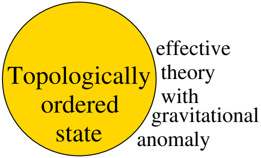
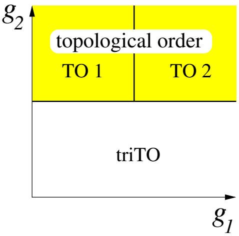
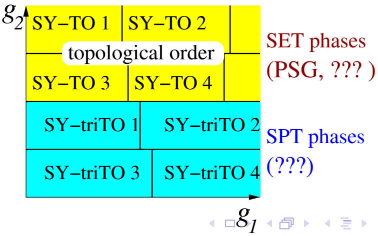






SPT order beyond $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$





CS-like topological terms $\leftrightarrow$ SPT and topo. orders

Pure SPT order within $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$: $\quad W^d_{\text{topinv}} = ac_1$

W^d_{topinv} only depend on A – the gauge G-connection

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topological order : $W^d_{\text{topinv}} = \omega_3, \frac{1}{2}\mathrm{w}_2\mathrm{w}_3$

W^d_{topinv} only depend on Γ – the gravitational SO-connection
p_1 is the first Pontryagin class, $\mathrm{d}\omega_3 = p_1$, and w_i is the Stiefel-Whitney classes.

[Kapustin 14]

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Topo. order w/ no topo. excitations

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- *$\mathbb{Z}_2$-class of 4+1D iTO's are generated by $\frac{1}{2}w_2w_3$.* [Kapustin 14]

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Mixed SPT order beyond $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$: $W^d_{\text{topinv}} = c_1\omega_3$

W^d_{topinv} depend on both A and Γ

A model to realize all (?) bosonic pure STP orders, mixed SPT orders, and invertible topological orders

- NLσM (group cohomology) approach to pure SPT phases:
  (1) NLσM+topo. term: $\frac{1}{2\lambda}|\partial g|^2 + 2\pi\,\mathrm{i}\,W(g^{-1}\partial g),\ g \in G$
  (2) Add symm. twist: $\frac{1}{2\lambda}|(\partial - \mathrm{i}A)g|^2 + 2\pi\,\mathrm{i}\,W[(\partial - \mathrm{i}A)g]$
  (3) Integrate out matter field: $Z_{\text{fixed}} = \mathrm{e}^{2\pi\,\mathrm{i} \int W_{\text{topinv}}(A)}$

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- $G \times SO_\infty$ NLσM (group cohomology) approach:
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  (2) Add twist: $\frac{1}{2\lambda}|(\partial - \mathrm{i} A - \mathrm{i}\Gamma)g|^2 + 2\pi \mathrm{i}\, W[(\partial - \mathrm{i} A - \mathrm{i}\Gamma)g]$
  (3) Integrate out matter field: $Z_{\text{fixed}} = \mathrm{e}^{2\pi \mathrm{i} \int W_{\text{topinv}}(A,\Gamma)}$

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  (3) Integrate out matter field: $Z_{\text{fixed}} = \mathrm{e}^{2\pi\mathrm{i}\int W_{\text{topinv}}(A,\Gamma)}$
- All possible topo. terms are classified by $\mathcal{H}^d(G \times SO, \mathbb{R}/\mathbb{Z})$
  Pure/mixed SPT orders, and iTOs are classified by
  $\mathcal{H}^d(G \times SO, \mathbb{R}/\mathbb{Z})$
  $= \mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) \oplus_{k=1}^{d-1} \mathcal{H}^k[G, \mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})$
  But not one-to-one. Need to quotient out something $\Gamma^d(G)$.

Trying to classify bosonic pure STP orders, mixed SPT orders, and invertible topological orders

- **Pure STP**: $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ one-to-one
- **Mixed SPT**: $\oplus_{k=1}^{d-1}\mathcal{H}^k[G, \mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})]$ many-to-one
- **iTO's**: $\mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})$ many-to-one

- Pure STP orders are classified by $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$
  CS-like topo. inv. $W^d_{\text{topinv}}(A)$ are classified by $H^{d+1}(BG, \mathbb{Z})$
  Pure STP orders are also classified by $W^d_{\text{topinv}}(A)$

- But iTOs are not one-to-one classified by $W^d_{\text{topinv}}(\Gamma_{SO})$ in $H^{d+1}(BSO, \mathbb{Z})$, because different $W^d_{\text{topinv}}(\Gamma_{SO})$ and $\tilde{W}^d_{\text{topinv}}(\Gamma_{SO})$ may satisfy $W^d_{\text{topinv}}(\Gamma_{SO}) = \tilde{W}^d_{\text{topinv}}(\Gamma_{SO})$ when Γ_{SO} is the SO-connection of the tangent bundle of M^d.

- W^d_{topinv}, $\tilde{W}^d_{\text{topinv}}$ $\to$ the same iTO $\to$ $\text{iTO}^d = \mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})/\Gamma^d$

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- **Pure STP orders**: $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ (the black entries below)
- **iTO's**: $\text{iTO}^d = \mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})/\Gamma^d$ (using Wu class and Sq^n)
- **Mixed SPT order** $\oplus_{k=1}^{d-1}\mathcal{H}^k(G, \text{iTO}^{d-k}) \subset \frac{\oplus \mathcal{H}^k[G, \mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})]}{\Gamma^d(G)}$

$G \setminus d =$	0+1	1+1	2+1	3+1	4+1	5+1	6+1
iTO^d	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	0
Z_n	$\mathbb{Z}_n$	0	$\mathbb{Z}_n$	0	$\mathbb{Z}_n \oplus \mathbb{Z}_n$	$\mathbb{Z}_{\langle n,2\rangle}$	$\mathbb{Z}_n \oplus \mathbb{Z}_n \oplus \mathbb{Z}_{\langle n,2\rangle}$
Z_2^T	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	0	$\mathbb{Z}_2 \oplus 2\mathbb{Z}_2$	$\mathbb{Z}_2$
$U(1)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$
$U(1) \rtimes Z_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus 2\mathbb{Z}_2$	$\mathbb{Z} \oplus 2\mathbb{Z}_2 \oplus \mathbb{Z} \oplus 2\mathbb{Z}_2$
$U(1) \times Z_2^T$	0	$2\mathbb{Z}_2$	0	$3\mathbb{Z}_2 \oplus \mathbb{Z}_2$	0	$4\mathbb{Z}_2 \oplus 3\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$U(1) \rtimes Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$	$2\mathbb{Z}_2 \oplus 2\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus 3\mathbb{Z}_2 \oplus \mathbb{Z}_2$

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$G \setminus d =$	0+1	1+1	2+1	3+1	4+1	5+1	6+1
iTO^d	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	0
Z_n	$\mathbb{Z}_n$	0	$\mathbb{Z}_n$	0	$\mathbb{Z}_n \oplus \mathbb{Z}_n$	$\mathbb{Z}_{\langle n,2\rangle}$	$\mathbb{Z}_n \oplus \mathbb{Z}_n \oplus \mathbb{Z}_{\langle n,2\rangle}$
Z_2^T	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	0	$\mathbb{Z}_2 \oplus 2\mathbb{Z}_2$	$\mathbb{Z}_2$
$U(1)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$
$U(1) \rtimes Z_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus 2\mathbb{Z}_2$	$\mathbb{Z} \oplus 2\mathbb{Z}_2 \oplus \mathbb{Z} \oplus 2\mathbb{Z}_2$
$U(1) \times Z_2^T$	0	$2\mathbb{Z}_2$	0	$3\mathbb{Z}_2 \oplus \mathbb{Z}_2$	0	$4\mathbb{Z}_2 \oplus 3\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$U(1) \rtimes Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$	$2\mathbb{Z}_2 \oplus 2\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus 3\mathbb{Z}_2 \oplus \mathbb{Z}_2$

- Probe mixed SPT order described by $\mathcal{H}^k[G, \mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})]$: put the state on $M^d = M^k \times M^{d-k}$ and add a G-symmetry twist on M^k → Induce a state on M^{d-k} described by $\mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})$ → a iTO state in iTO^{d-k}

A mechanism for Z_2^T mixed SPT state in 3+1D

[Vishwanash-Senthil 12, Kapustin 14, Wen 14]

The Z_2^T mixed SPT states are classified by
$\mathcal{H}^1(Z_2^T, \mathrm{iTO}^3) = \mathbb{Z}_2$

The topological invariant for a Z_2^T mixed SPT state (bosonic topological super fluid with time reversal symmetry) is
$W^4_{\text{topinv}} = \frac{1}{2} p_1$ [W 14] ($W^4_{\text{topinv}} = \frac{1}{6} p_1$ [VS 12, K 14])

- Start with a T-symmetry breaking state. Proliferate the symmetry breaking domain walls to restore the T-symmetry.
  $\to$ a trivial SPT state.
- Bind the domain walls to $(E_8)^3$ [W 14] (E_8 [VS 12, K 14]) quantum Hall state, and then proliferate the symmetry breaking domain walls to restore the T-symmetry.
  $\to$ a non-trivial Z_2^T SPT state.

Z_2 SPT phases and their physical properties

- Topological terms:

$$\oint A_{Z_2} = 0, \pi; \; a_1 \equiv \frac{A_{Z_2}}{\pi};$$

$d =$	$\mathcal{H}^d[Z_2]$	W^d_{topinv}
$0+1$	$\mathbb{Z}_2$	$\frac{1}{2}a_1$
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  $= \mathrm{e}^{\mathrm{i}\, k\pi \oint_{S^1} a_1} = \mathrm{e}^{\mathrm{i}\, k\pi} = \pm 1$

  $\to$ ground state Z_2-charge: $k = 0, 1$

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- **In $2+1$D**, $\int_{M^3} W^3_{\text{topinv}} = \int_{M^3} \frac{1}{2} a_1^3$.
  Here we do not view a_1 as 1-form
  but as 1-cocycle $a_1 \in H^1(M^3, \mathbb{Z}_2)$, and $a_1^3 \equiv a_1 \cup a_1 \cup a_1$:
  $\int_{M^3} a_1 \cup a_1 \cup a_1 = 0$ or $1 \to \mathrm{e}^{2\pi\,\mathrm{i}\oint_{M^3} W_{\text{topinv}}} = \mathrm{e}^{\pi\,\mathrm{i}\oint_{M^3} a_1^3} = \pm 1$

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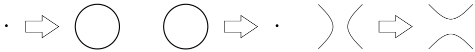
  $\int_{M^3} a_1 \cup a_1 \cup a_1 = 0 \text{ or } 1 \to \mathrm{e}^{2\pi \mathrm{i} \oint_{M^3} W_{\text{topinv}}} = \mathrm{e}^{\pi \mathrm{i} \oint_{M^3} a_1^3} = \pm 1$
- Poincaré duality: 1-cocycle $a_1 \leftrightarrow$ 2-cycle N^2 (2D submanifold)
  N^2 is the surface across which we do the $\mathbb{Z}_2$ symmetry twist.
  Choose $M^3 = M^2 \rtimes S^1$
  As we go around S^1:

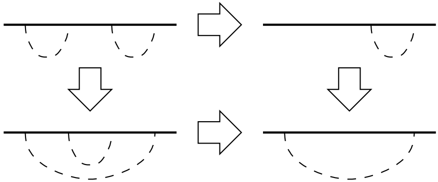


(a) (b) (c)

$\int_{M^3} a_1^3 = \#$ of loop creation/annihilation $+ \#$ of line reconnection

The edge of Z_2 SPT phase must be gapless or symmetry breaking Chen-Liu-Wen 11; Levin-Gu 12

Assume the edge of a Z_2 SPT phase is gapped with no symmetry breaking. We use Z_2 twist try to create excitations (called Z_2 domain walls) at the edge. We may naively expect those Z_2-domain walls are trivial, but they are not. They have a non-trivial fusion property: different fusion order can differ by a $-$ sign.



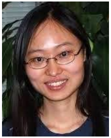
So the Z_2 domain walls on the boundary form a non-trivial fusion category.

$\to$ the bulk state must carry a non-trivial topological order.

The boundary of the 2+1D Z_2 SPT state has a 1+1D bosonic global Z_2 anomaly [Chen-Wen 12]

The 1+1D bosonic global Z_2 anomaly $\to$ The edge of Z_2 SPT phase must be gapless or symmetry breaking.

- One realization of the edge is described by 1+1D XY model or $U(1)$ CFT. The primary field (vertex operator) $V_{l,m}$ has dimensions $(h_R, h_L) = (\frac{(l+2m)^2}{8}, \frac{(l-2m)^2}{8})$.



- The Z_2 symmetry action $V_{l,m} \to (-)^{l+m} V_{l,m}$

Such a 1+1D Z_2 symmetry is anomalous:

(1) The XY model has no UV completion in 1+1D such that the Z_2 symmetry is realized as an on-site symm. $U = \prod_i \sigma_i^x$.
(2) If we gauge the Z_2, the 1+1D Z_2 gauge theory has no UV completion in 1+1D as a bosonic theory.
(3) The XY model has a UV completion as boundary of 2+1D lattice theory w/ the Z_2 symmetry realized as an on-site symm.