Is spacetime a quantum error-correcting code?

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= *HaPPY*

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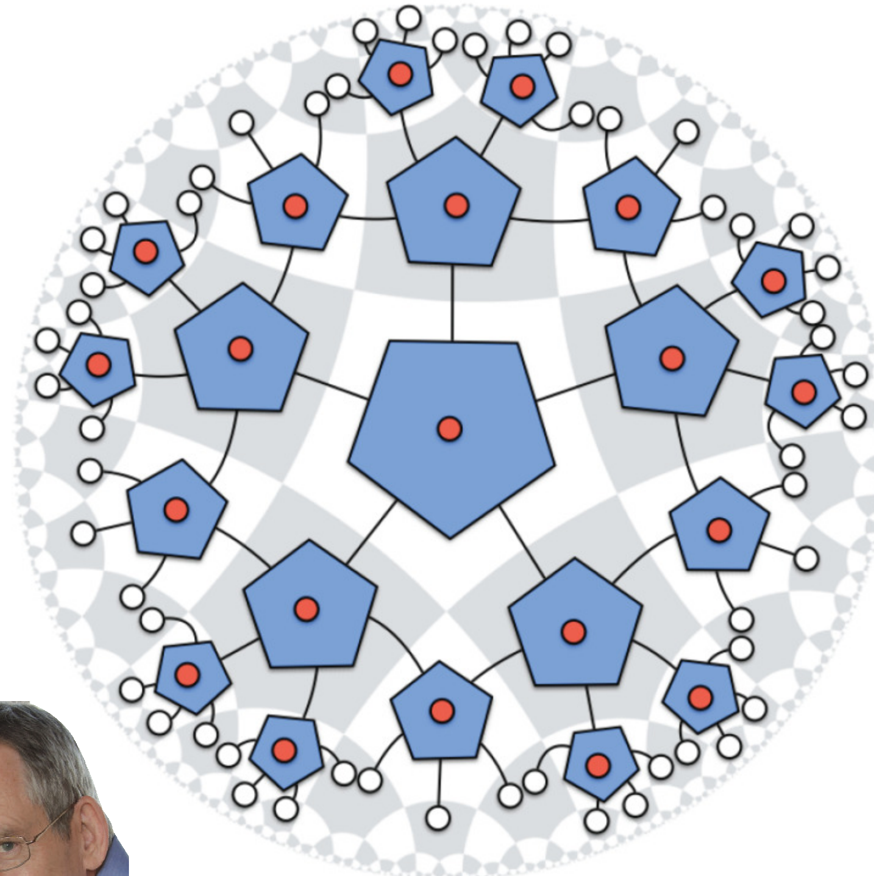





Two amazing ideas

Holographic principle

Quantum error correction

Both from the mid-1990s, and (maybe) closely related.

Bulk/boundary duality: an *exact* correspondence

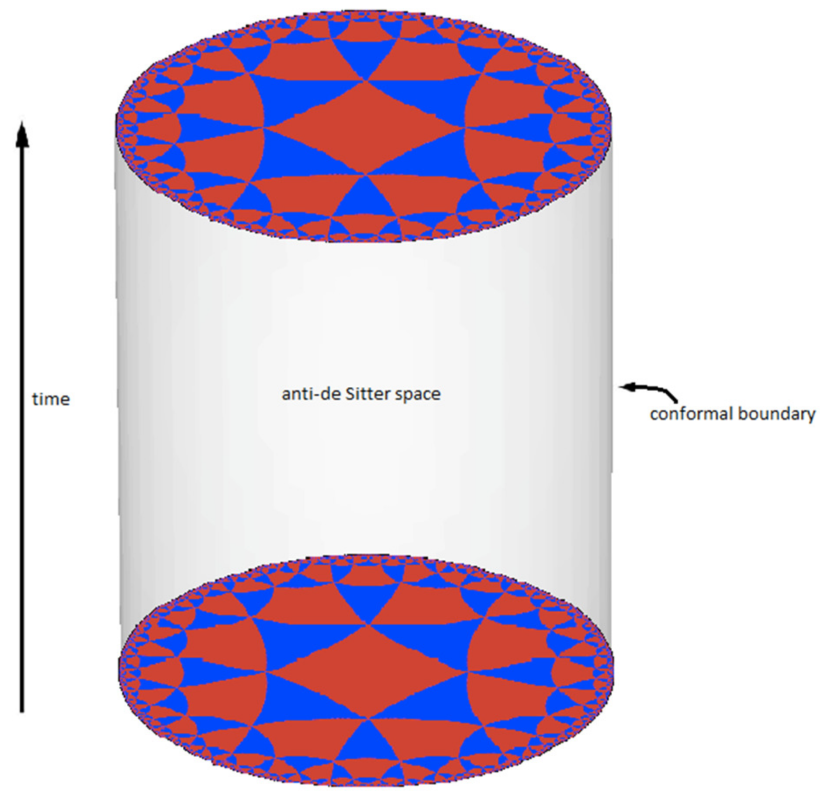
Weakly-coupled gravity in the bulk $\leftrightarrow$ strongly-coupled conformal field theory on boundary.

Complex dictionary maps bulk operators to boundary operators.

Emergent radial dimension can be regarded as an RG scale.

Semiclassical (sub-AdS scale) bulk locality is highly nontrivial.

Geometry in the bulk theory is related to entanglement structure of the boundary theory.



"AdS3 (new)" by Polytope24 - Own work.
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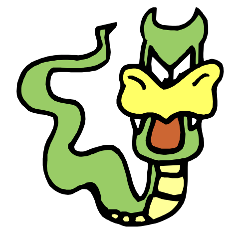
Recovering from error



$$|\psi\rangle\otimes|0\rangle_{Env}\otimes|0\rangle_{Anc} \xrightarrow{\text{error}} \sum_a E_a|\psi\rangle\otimes|a\rangle_{Env}\otimes|0\rangle_{Anc}$$

$$\xrightarrow{\text{recover}} \sum_a |\psi\rangle\otimes|a\rangle_{Env}\otimes|a\rangle_{Anc} = |\psi\rangle\otimes|\varphi\rangle_{Env-Anc}$$

Errors entangle the data with the environment (*decoherence)*.

Recovery transforms entanglement of data with environment into entanglement with ancilla (which can be discarded), purifying data.

Data cools as ancilla heats (*dissipative process*, requiring power).

For this to work:

-- errors must be or a restricted type, e.g. with support on a small fraction of the physical qubits.

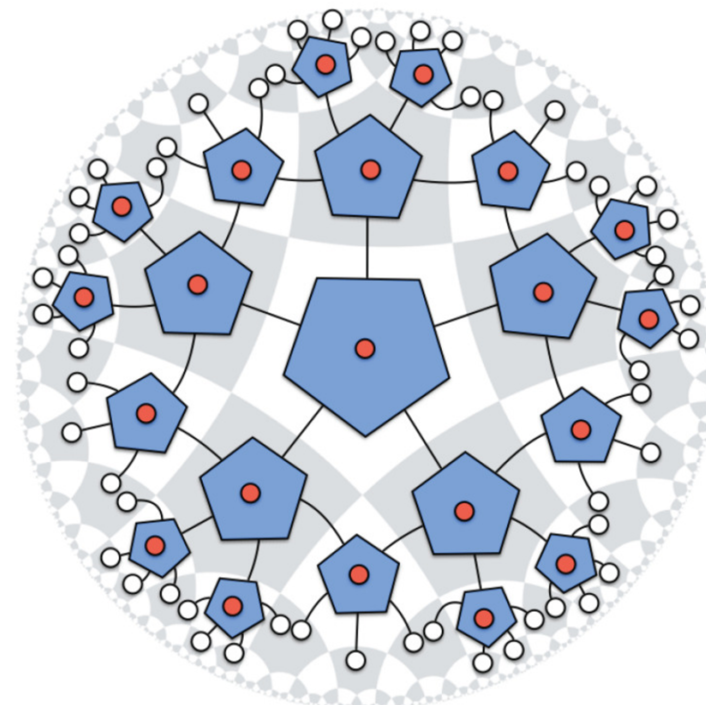
-- the protected state belongs to a *quantum code*, an appropriately chosen subspace of the physical Hilbert space. (No action of errors on code space.)

Toy model for bulk/boundary correspondence

Holographic codes demonstrate the idea that *geometry emerges from entanglement*.

A tensor network realization of holography, based on a uniform tiling of the bulk. (Lattice spacing comparable to AdS curvature scale. No dynamics.)

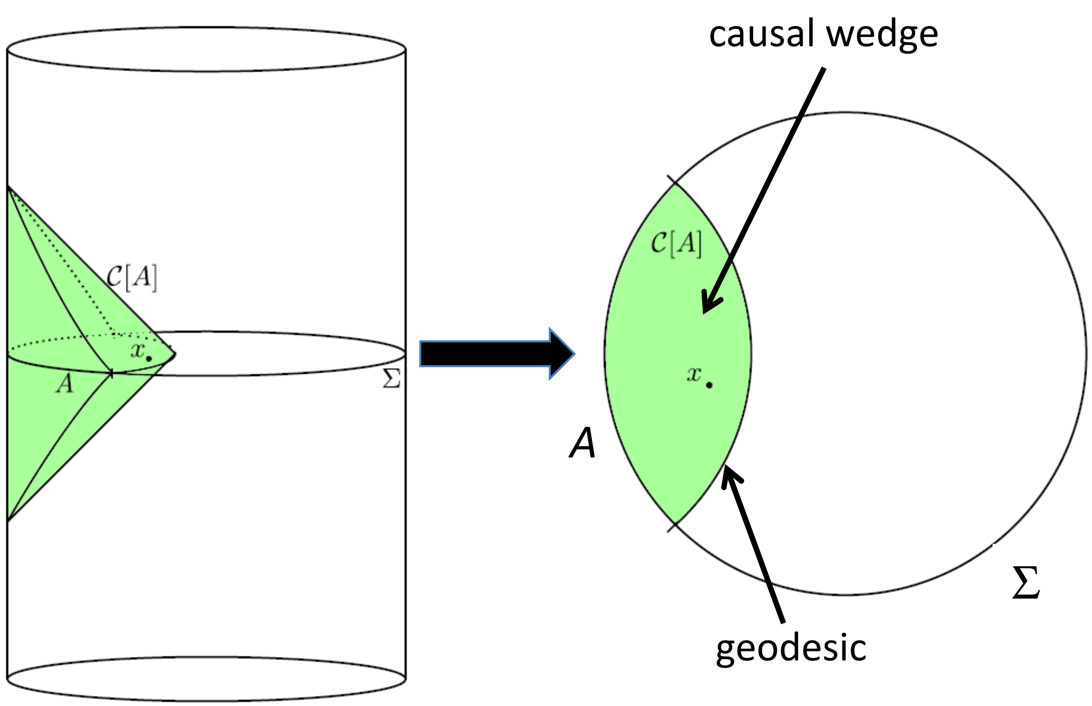
Physical variables of a quantum code reside on the boundary, logical operators (those preserving the code space) reside in the bulk.



There is an explicitly computable dictionary, and computable boundary entanglement structure. Local operators deep in the bulk are mapped to highly nonlocal operators on the boundary.

This dictionary is not complete --- the bulk Hilbert space (code space) is a proper subspace of the boundary Hilbert space, and the bulk operators preserve this subspace. E.g. we may think of them as operators which map low-energy states to low-energy states in the boundary CFT.

AdS-Rindler reconstruction



Bulk time slice contains point *x* in the bulk and boundary time slice Σ.

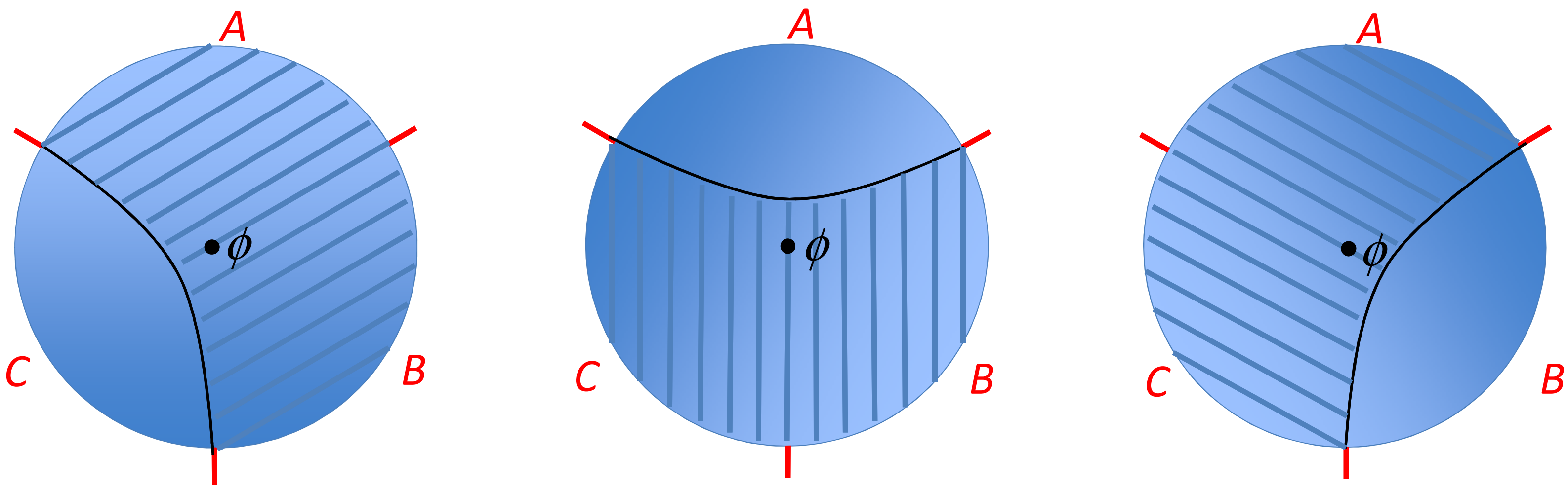
A local operator acting at *x* can be reconstructed on the boundary region *A* if *x* lies within the *causal wedge C[A]* of *A*, the bulk region bounded by *A* and the bulk geodesic with the same boundary as *A*.

Classical bulk field equations are "causal" in the *radial* direction. Boundary data suffices to determine a bulk operator at *x* if *x* lies within the "backward" light cone of *x* (*Hamilton-Kabat-Lifschytz-Lowe*). This can be systematically corrected order by order in 1/N.

Furthermore we can use the boundary equations to squash the boundary wedge down to the time slice Σ.

This reconstruction is highly ambiguous – each bulk point lies in many causal wedges.

Causal wedge puzzle



The ambiguity of the AdS-Rindler reconstruction poses an interesting puzzle.

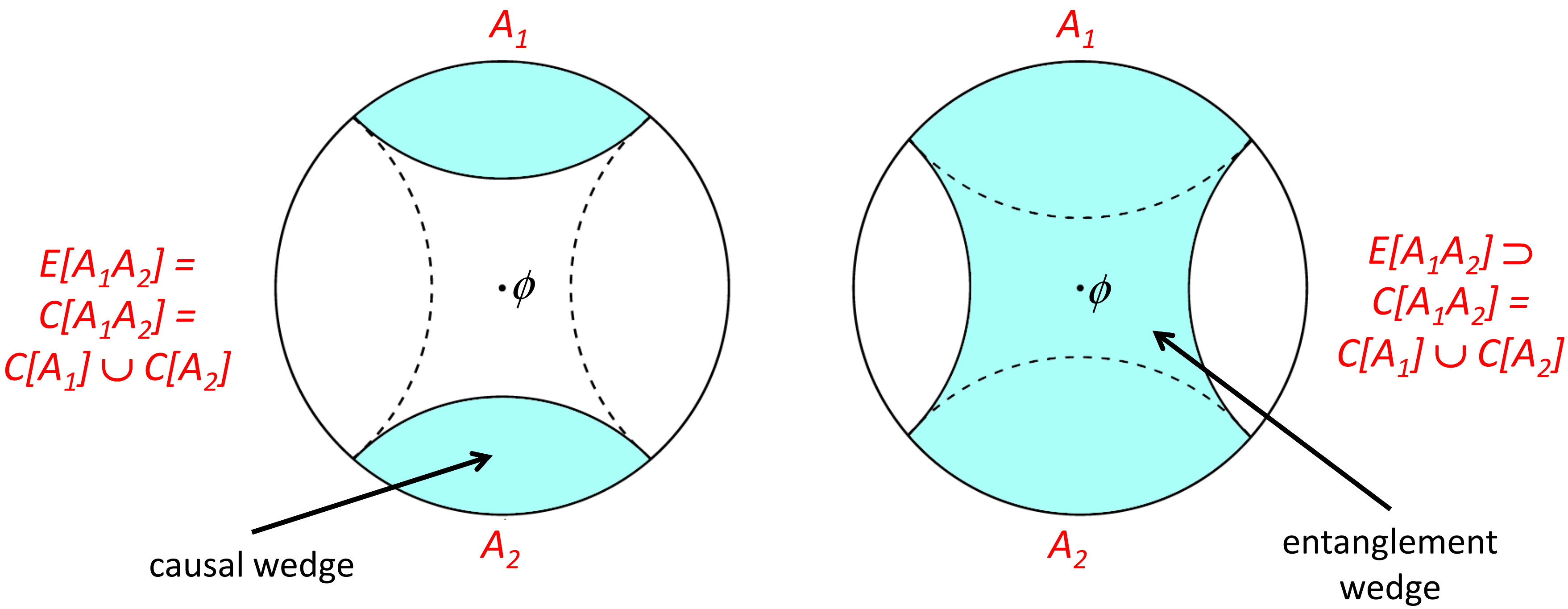
Divide boundary into three disjoint sets *A, B, C* as shown. The bulk operator ϕ resides in *C[AB],* hence can be reconstructed on *AB* and commutes with any boundary operator in *C*.

We can also reconstruct ϕ on *BC* (so it commutes with operators in *A*) or on *AC* (so it commutes with operators in *B*). Hence the reconstructed operator commutes with all local boundary operators, and must be a multiple of the identity (local field algebra is irreducible).

Resolution (*Almheiri-Dong-Harlow*): These three reconstructions yield physically inequivalent boundary operators, all with the same action on the code subspace. *Holographic codes concretely realize this proposal.*

Alternative viewpoint (*Mintun-Polchinski-Rosenhaus*): Appeal to gauge invariance of the boundary theory. Either way, *redundancy* in boundary operators provides protection against erasure of a (sufficiently small) portion of the boundary.

Entanglement wedge vs. causal wedge

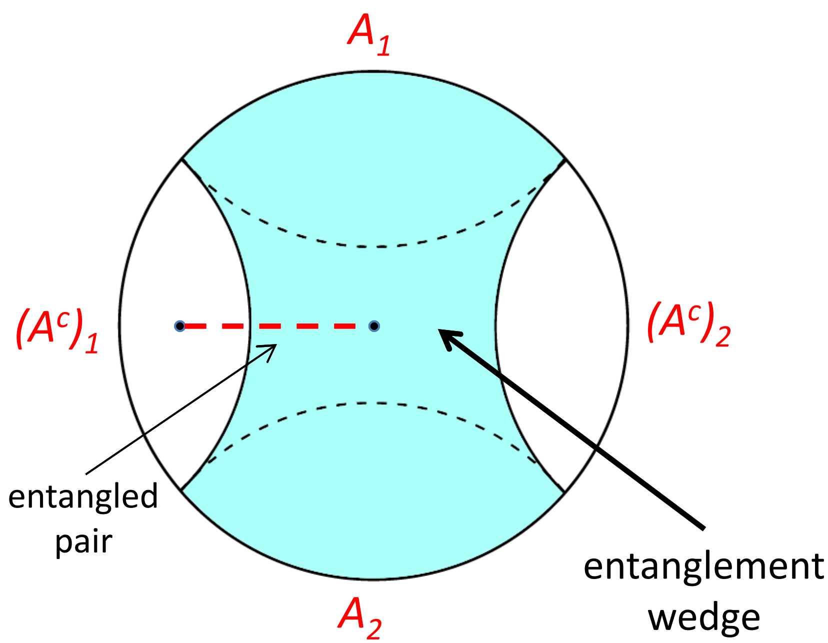


Conjecture: Bulk operators residing in the *entanglement wedge E(A)* of *A* can be reconstructed on boundary region *A (Wall, Headrick-Hubeny-Lawrence-Rangamani)*.

If *A* is a union of two or more disconnected components, then *E(A)* may extend into the bulk far beyond the causal wedge *C(A).*

In holographic codes, bulk operators far beyond the causal wedge of A can be reconstructed on A when A is a suitable disconnected region.

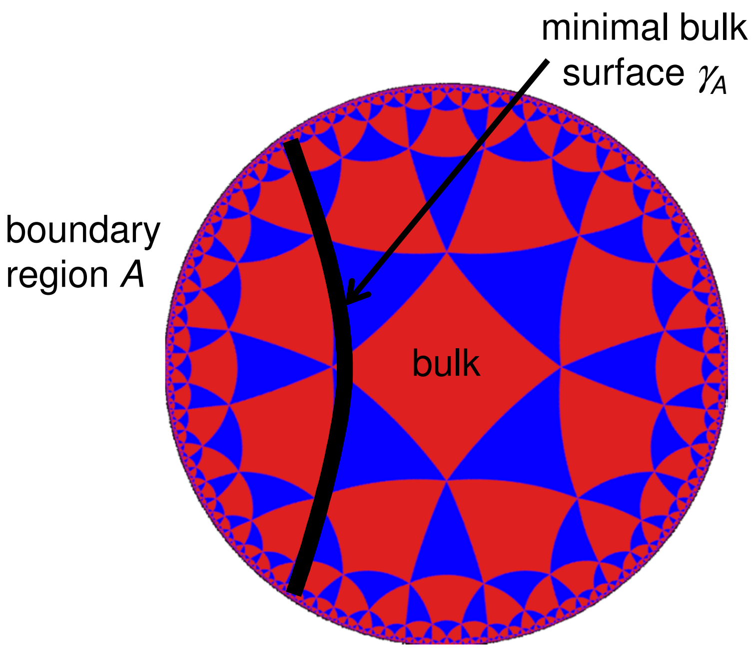
Entanglement wedge conjecture



Entangled pairs in the bulk, where one member of the pair is inside *E(A)* and the other is outside, contribute to the entropy of *A*, i.e. to the entanglement shared by *A* and its complement A^c (*Faulkner-Lewkowycz-Maldacena*).

Therefore, there should be operators in *A* which can detect the member of the pair inside *E(A).*

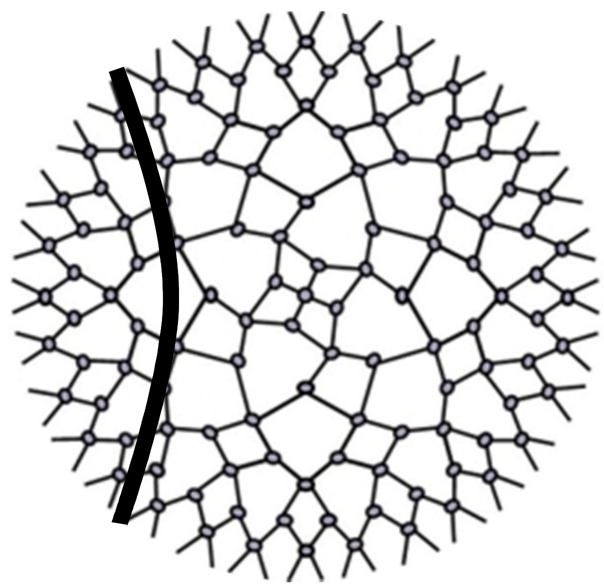
Holographic entanglement entropy



$$S(A) = \frac{1}{4G_N}\,\mathrm{Area}(\gamma_A) + \cdots$$

To compute entropy of region *A* in the boundary field theory, find minimal area of the bulk surface γ_A with the same boundary (*Ryu-Takayanagi*).

Similar hyperbolic geometry is realized by MERA tensor networks, which therefore also have entropy bounded above by the logarithm of the size of A (*Swingle*).



In holographic codes, the Ryu-Takayanagi is *exact* in certain cases.

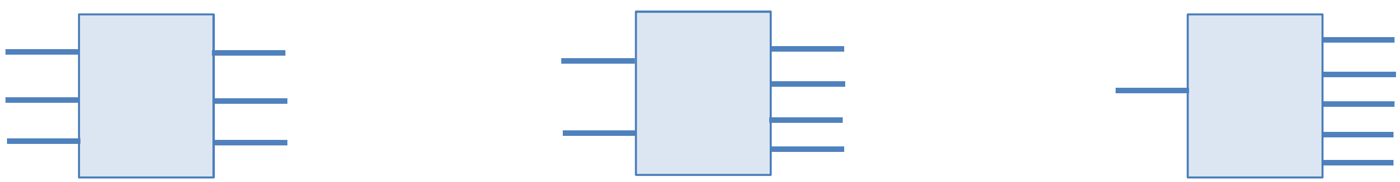
Perfect tensors

The tensor *T* arises in the expansion of a pure state of 2*n* *v*-dimensional "spins" in an orthonormal basis.

$$|\psi\rangle = \sum_{a_1, a_2, \ldots, a_{2n}} T_{a_1 a_2 \ldots a_{2n}} \, |a_1 a_2 \ldots a_{2n}\rangle$$

T is perfect if the state is maximally entangled across *any* cut, i.e. for any partition of the 2*n* spins into two sets of *n* spins.

By transforming kets to bras, *T* also defines 3→3 unitary, 2→4 and 1→5 isometries.



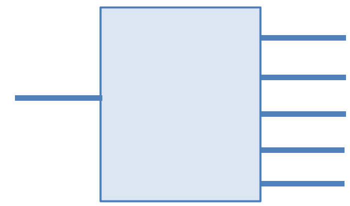
$$\sum_{a_1, \ldots a_6} T_{a_1 \ldots a_6} \, |a_4 a_5 a_6\rangle\langle a_1 a_2 a_3| \qquad \sum_{a_1, \ldots a_6} T_{a_1 \ldots a_6} \, |a_3 a_4 a_5 a_6\rangle\langle a_1 a_2| \qquad \sum_{a_1, \ldots a_6} T_{a_1 \ldots a_6} \, |a_2 a_3 a_4 a_5 a_6\rangle\langle a_1|$$

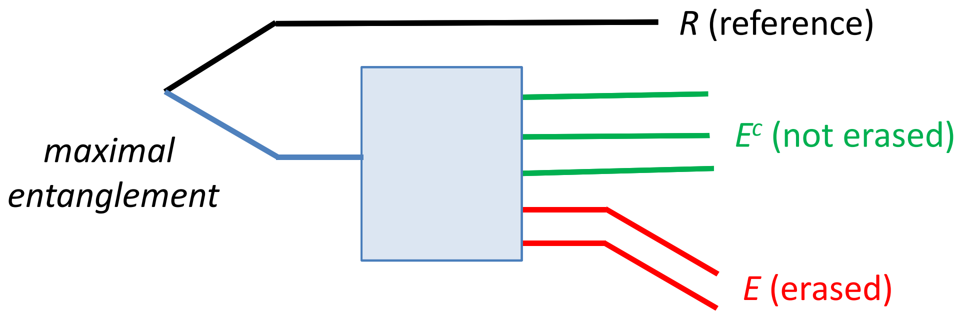
These are the isometric encoding maps (up to normalization) of quantum error-correcting codes. The 2→4 map encodes two qubits in a block of 4, and corrects 1 erasure. The 1→5 map encodes one qubit in a block of 5, and corrects 2 erasures.

Erasure correction

The 1→5 isometric map encodes one qubit in a block of 5, and corrects two erasures.

$$\sum_{a_1,\ldots a_6} T_{a_1\ldots a_6} \,|\, a_2 a_3 a_4 a_5 a_6 \rangle \langle a_1 \,|$$





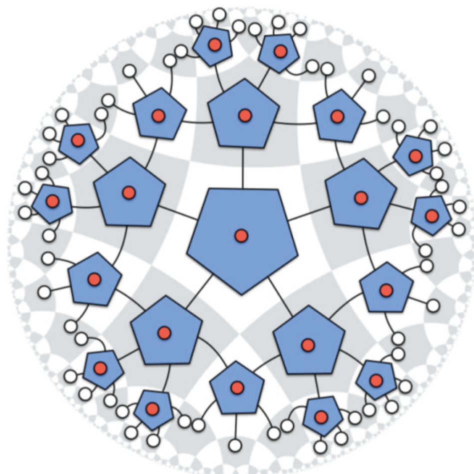
We say qubits are erased if they are removed from the code block. But we know *which* qubits were erased and may use that information in recovering from the error.

Consider maximally entangling a *reference qubit R* with the encoded qubit. Suppose two physical qubits (the subsystem *E*) are removed, while their complement E^c is retained.

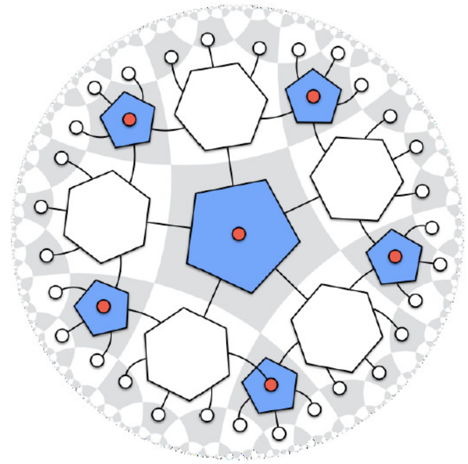
Because the tensor *T* is perfect, *RE* is maximally entangled with E^c, hence *R* is maximally entangled with a subsystem of E^c. Thus the logical qubit can be decoded by applying a unitary decoding map to E^c alone; *E* is not needed.

Likewise, we may apply any logical operator to the encoded qubit by acting on E^c alone. (The logical operation can be *cleaned* so it has no support on the erased qubits.)

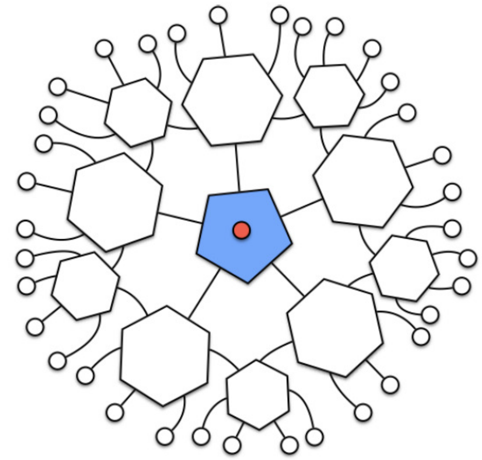
Holographic quantum codes



pentagon code



pentagon/hexagon code



one encoded qubit

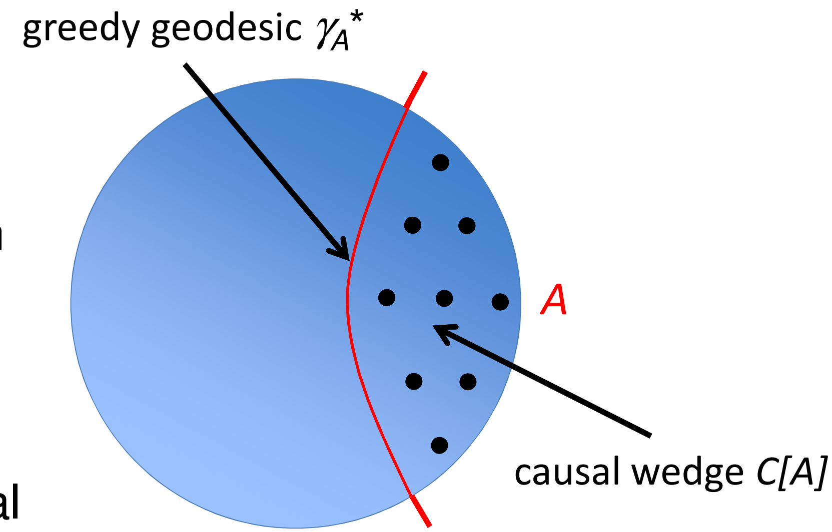
Holographic quantum error-correcting codes are constructed by contracting perfect tensors according to a tiling of hyperbolic space by polygons.

There are two types of uncontracted indices: bulk indices and boundary indices. These are not separate subsystems; rather the code is defined by an isometric embedding of the bulk Hilbert space into the boundary Hilbert space, obtained by composing the isometries associated with each perfect tensor.

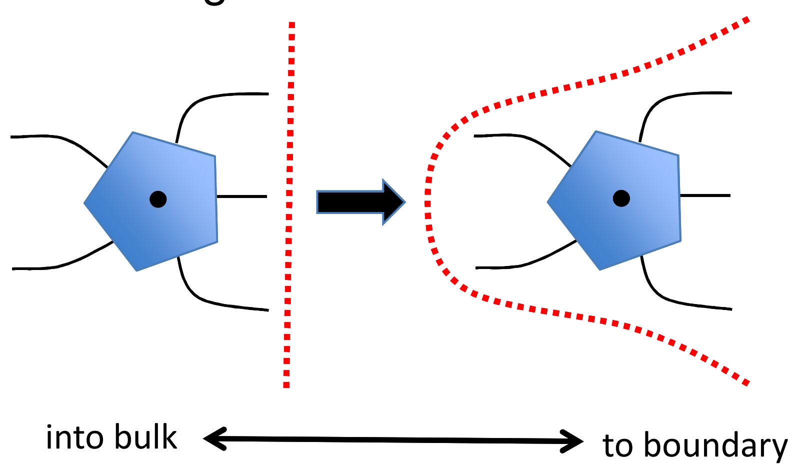
E.g., start at the central pentagon and contract successive layers. Each pentagon/hexagon has at most two incoming indices from the previous layer and at most one bulk index; hence provides an isometry from incoming and bulk indices to outgoing indices.

“Greedy” causal wedge

There is an analog of the AdS-Rindler reconstruction in holographic codes. For a connected region *A* on the boundary there is a corresponding *greedy geodesic* γ_A^* and *greedy causal wedge C[A].* Bulk operators contained in *C[A]* can be reconstructed on *A*.



A given bulk operator is contained in many different causal wedges; it is protected against erasure of the physical qubits outside the causal wedge. Operators deeper in the bulk have better protection against erasure.

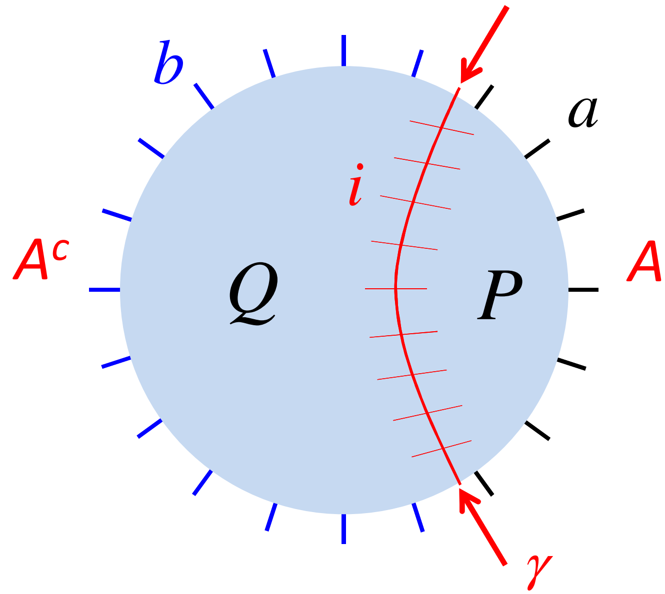


To construct the greedy geodesic, start with *A*, and push into the bulk a cut bounded by ∂A, step by step: if at least three of the tensor’s legs cross the cut, move the cut further into the bulk past the tensor. After each step we have an isometry mapping indices which cross the cut (and bulk indices) to *A*.

The greedy geodesic coincides with true geodesic (minimal cut) in some cases, differs slightly in other cases.

Ryu-Takayanagi Formula

Consider a *holographic state* $|\ \psi\rangle$ (no dangling bulk indices), and a cut γ through the bulk with indices on the cut labeled by i. Indices of *A* are labeled by a and indices of A^c labeled by b.

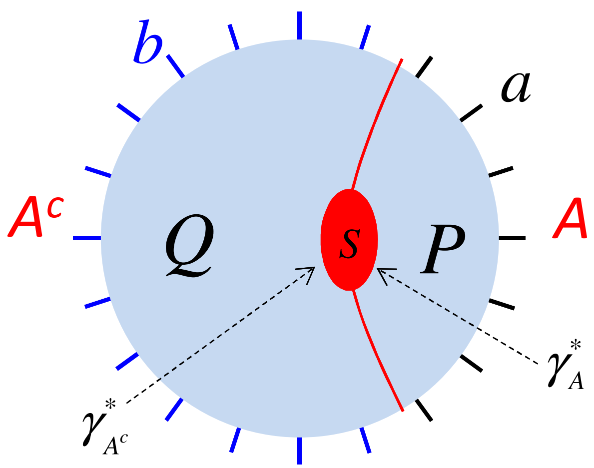


$$|\psi\rangle = \sum_{a,b,i} |a\rangle_A \otimes |b\rangle_{A^c} P_{ai}\ Q_{bi} = \sum_i |P_i\rangle_A \otimes |Q_i\rangle_{A^c}$$

There are v^γ terms in the sum over i. If the tensors P and Q are isometries (up to normalization), then the vectors $\{\ |\ P\ \rangle_i\ \}, \{\ |\ Q\ \rangle_i\ \}$ are orthonormal.

For a holographic state on a tiling with *nonpositive curvature*, the greedy geodesics of *A* and A^c match, if *A* is connected (*max-flow min-cut argument*).

Thus P and Q *are* both isometries, and:



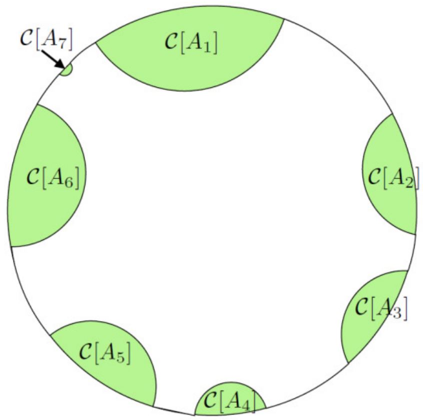
$$S(A) = |\gamma_A| \log v$$

In other cases, the two greedy geodesics might not match exactly; an additional tensor is trapped inside a "residual region" between them, and there are O(1) corrections to the Ryu-Takayanagi formula.

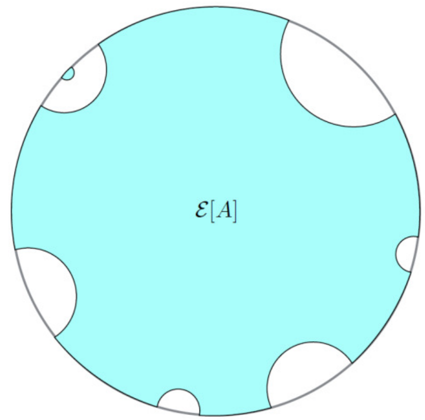
Entanglement wedge vs. causal wedge

The entanglement wedge reaches far beyond the causal wedge if the region *A* has many connected components, with small gaps between them.

Easier to analyze: i.i.d erasure of physical boundary qubits (erasure probability p). Can logical operations deep in the bulk be reconstructed on the unerased part of the boundary?



shallow causal wedge

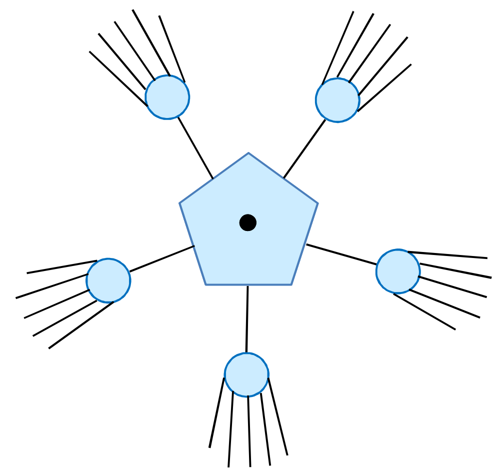


deep entanglement wedge

Still easier: *concatenated code* (tree graph). Central qubit is encoded in a block of 5, each of these is encoded in a block of 5, etc.

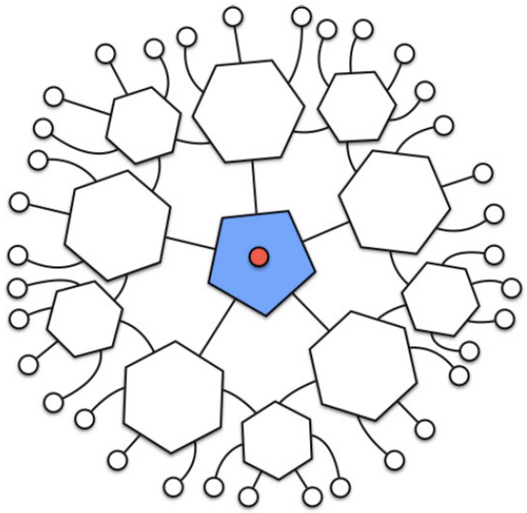
To recover, start from physical boundary qubits (level 0). For a level-($j+1$) qubit to be erased, three must be erased at level j.

$$p_{j+1} \leq 10 p_j^3 = p_c \left(\frac{p_j}{p_c}\right)^3 \Rightarrow p_j \leq p_c \left(\frac{p}{p_c}\right)^{3^j} \text{ where } p_c = \frac{1}{\sqrt{10}} = .316$$



concatenated code

Erasure threshold



one encoded qubit

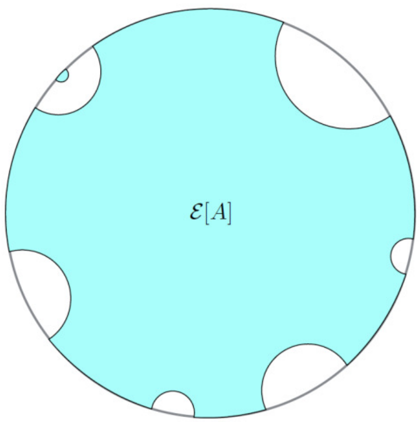
The analysis for holographic codes is harder because the graph is not a tree, but we can still do a hierarchical analysis of the failure probability.

The main complication is that a hexagon at level j may be contracted with two different hexagons at level $j+1$. Therefore, the noise is no longer i.i.d. beyond level 0, and we need to deal with noise correlations. This is not so bad, because the hyperbolic geometry controls the spread of the correlations (noise correlations beyond nearest neighbors never arise at any level of the hierarchy).

$p_j \le p_c \left(\frac{p_j}{p_c}\right)^{\lambda^j}$ where $\lambda = \frac{1+\sqrt{5}}{2} = 1.618$ and $p_c = \frac{1}{12} = .083$

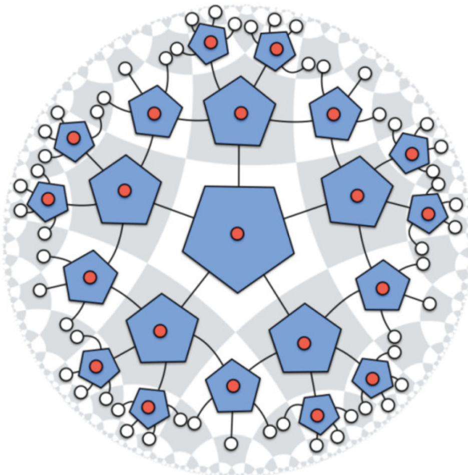
Numerics supports the expectation that p_c = ½.

For the pentagon/hexagon code with one encoded qubit, bulk operators acting on the central qubit can be reconstructed (with high probability) on a randomly chosen subset containing slightly more than half the boundary qubits.



deep entanglement wedge

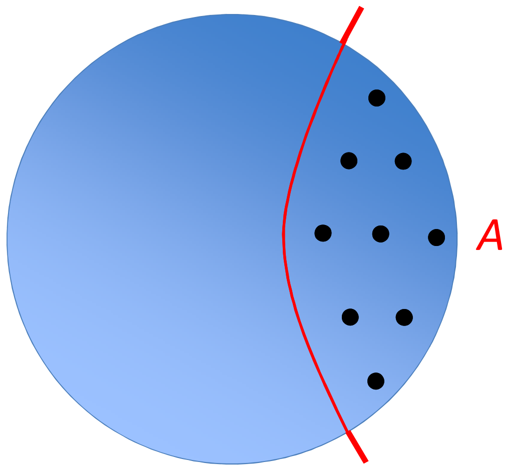
Holographic quantum codes



Nicely capture some central features of full blown gauge/gravity duality, and provide an explicit dictionary relating bulk and boundary observables.

Illustrate how quantum error correction resolves the causal wedge puzzle, and how the operators deep in the entanglement wedge can be reconstructed.

Realize exactly the Ryu-Takayanagi relation between boundary entanglement and bulk geometry (with small corrections in some cases).



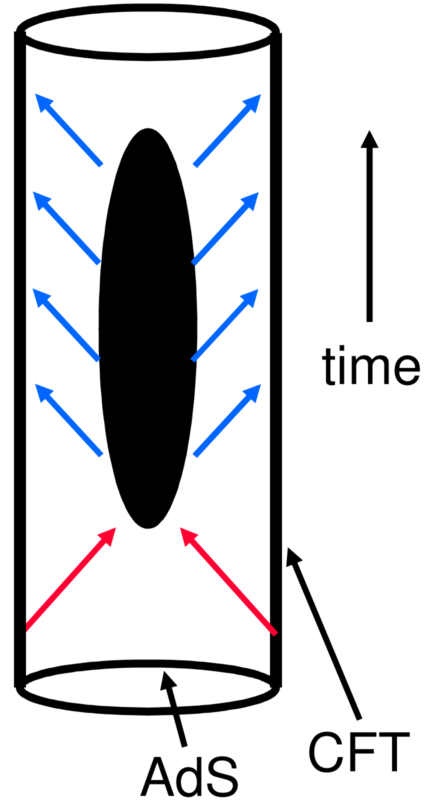
Allow flexibility in choice of lattice (including uniform lattices with discrete scale invariance) and of bulk operator algebra. Works in higher dimensions.

But … so far these models are not dynamical, and do not address bulk locality at sub-AdS distance scales.

Holographic codes may have other applications besides quantum gravity, e.g. quantum matter or fault-tolerant quantum computing.

Additional Slides

A black hole in a bottle



We can describe the formation and evaporation of a black hole using an “ordinary” quantum theory on the walls of the bottle, where information has nowhere to hide (*Maldacena*).

A concrete realization of the “holographic principle” (*’t Hooft, Susskind*).

So at least in the one case where we think we understand how quantum gravity works, a black hole seems not to destroy information!

Even so, the mechanism by which information can escape from behind a putative event horizon remains murky.

Indeed, it is not clear whether or how the boundary theory describes the experience of observers who cross into the black hole interior, or even if there is an interior!

Logical operator = precursor

A local operator in the bulk produces a disturbance which propagates causally in the bulk, producing a locally detectable signal on the boundary at a later time.

By evolving backward in time on the boundary, we obtain the nonlocal boundary *precursor operator* corresponding to the bulk local operator.

The precursor becomes more and more nonlocal for bulk operators deeper and deeper in the bulk.

We interpret the precursor as the logical operator of a quantum code, with better protection against error for bulk operators deeper inside the bulk.