- Agenda: explore **RSA** [8] via
  1. an "in theory", i.e., design-oriented perspective, and
  2. an "in practice", i.e., implementation-oriented perspective,
     - 2.1 multi-precision (or "big") integer arithmetic,
     - 2.2 exponentiation,
     - 2.3 modular mulitiplication.
- Caveat!

$\sim$ 2 hours $\Rightarrow$ introductory, and (very) selective (versus definitive) coverage.

Part 1: in theory (1)

- Recall: the RSA primitive can be used as
  - an asymmetric encryption scheme
    1. generation of a public and private key pair: given a security parameter λ

$$\mathsf{RSA.KeyGen}(\lambda) = \begin{cases} \text{select random } \frac{\lambda}{2}\text{-bit primes } p \text{ and } q \\ \text{compute } N = p \cdot q \\ \text{compute } \Phi(N) = (p-1) \cdot (q-1) \\ \text{select random } e \in \mathbb{Z}_N^* \text{ such that } \gcd(e, \Phi(N)) = 1 \\ \text{compute } d = e^{-1} \pmod{\Phi(N)} \\ \text{return public key } (N, e) \text{ and private key } (N, d) \end{cases}$$

    2. encryption of a plaintext $m \in \mathbb{Z}_N^\times$:

$$c = \mathsf{RSA.Enc}((N, e), m) = m^e \pmod{N}$$

    3. decryption of a ciphertext $c \in \mathbb{Z}_N^\times$:

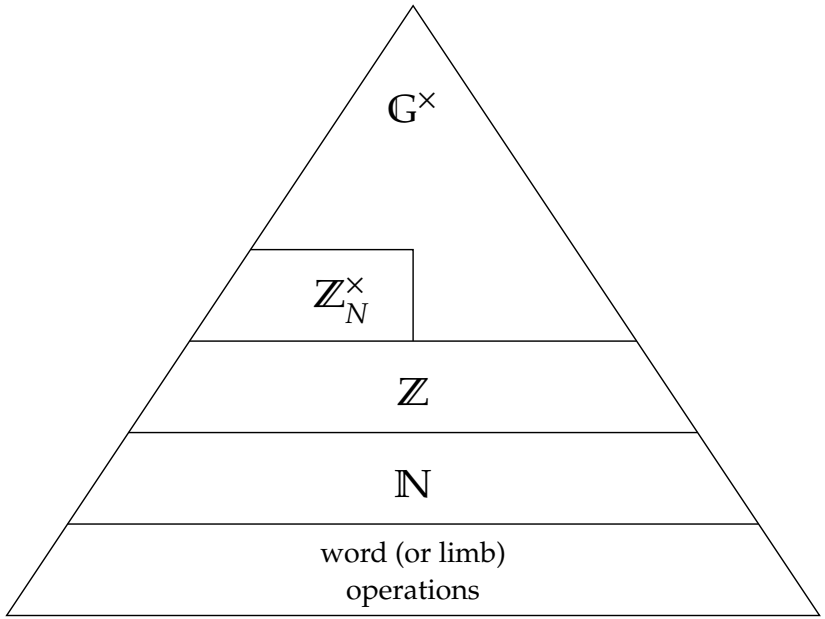
$$m = \mathsf{RSA.Dec}((N, d), c) = c^d \pmod{N}$$

  - a digital signature scheme: (very) roughly, we just set

$$\begin{aligned} \mathsf{RSA.Sig} &\simeq \mathsf{RSA.Dec} \\ \mathsf{RSA.Ver} &\simeq \mathsf{RSA.Enc} \end{aligned}$$

Part 2: in practice (1)

- Challenge:
  - given a functionality "stack", i.e.,



  bridge gap between what we have (bottom) and want (top),

  - an **implementation strategy** for doing so must consider many
    - goals : parameter set, functionality, ...
    - metrics : latency, throughput, memory footprint, ...
    - constraints : hardware versus software, data-path width, ...

    :

Part 2.1: in practice (1)

- **Problem**:
  - we want to represent and perform operations on integers, i.e., elements of
$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, +1, +2, +3, \ldots\},$$
  - a micro-processor equipped with a w-bit word size approximates $\mathbb{Z}$ using an appropriate data type; e.g., in C we get
$$\begin{array}{lllllllllllll} w & = & 8 & \rightsquigarrow & \texttt{char} & \simeq & \texttt{int8\_t} & \mapsto & \{ & -2^7, & \ldots, & 0, & \ldots, & +2^7 - 1 & \} \\ w & = & 16 & \rightsquigarrow & \texttt{short} & \simeq & \texttt{int16\_t} & \mapsto & \{ & -2^{15}, & \ldots, & 0, & \ldots, & +2^{15} - 1 & \} \\ w & = & 32 & \rightsquigarrow & \texttt{int} & \simeq & \texttt{int32\_t} & \mapsto & \{ & -2^{31}, & \ldots, & 0, & \ldots, & +2^{31} - 1 & \} \end{array}$$
  - the magnitude of some $x \in \mathbb{Z}$ can be much larger than a w-bit word.
- **Solution**:
  1. a data structure to represent x, and
  2. algorithms to operate on instances of said data structure.

Part 2.1: in practice (2)

- Idea:
  1. represent x using a base-b expansion

$$\begin{aligned} \hat{x} &= \langle \hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{n-1} \rangle \\ &\mapsto x \\ &= \pm \sum_{i=0}^{n-1} \hat{x}_i \cdot b^i \end{aligned}$$

     where each $\hat{x}_i \in X = \{0, \ldots, b-1\}$,

  2. select $b = 2^w$ such that
     - each $\hat{x}_i$ is termed a **limb**,
     - we can represent an $x \gg 2^w$, *and*
     - digits of x can be dealt with conveniently by the processor.

Part 2.1: in practice (3)

- Idea:
  1. represent $x \in \{0, 1, \ldots 2^w - 1\}$ using an instance of

     ```
     typedef uint32_t limb_t;
     ```

  2. represent $x \in \mathbb{N}$ using an array

     ```
     limb_t x[ l_x ]
     ```

     or a pointer to such an array, e.g.,

     ```
     limb_t* x
     ```

     plus an associated length `l_x`.

  3. represent $x \in \mathbb{Z}$ using an instance of

Listing

```
1 typedef struct __mpz_t {
2   limb_t d[ MPZ_LIMB_MAX ];
3
4   int    l;
5   int    s;
6 } mpz_t;
```

where

- `x.d` is a fixed-size array of limbs representing the magnitude of x,
- `x.l` is the number limbs used within `x.d`, and
- `x.s` is the sign of x.

Part 2.1: in practice (5)

Algorithms for $x \in \{0, 1, \ldots, 2^w - 1\}$

- Step #1: limb-focused operations, e.g., "multiply-accumulate with carry".
  - the idea is to support computation of

$$r_1 \cdot 2^w + r_0 = t \leftarrow e \cdot f + g + h,$$

  - for $b = 2^w$, the result r has at most $2 \cdot w$ bits because

$$(2^w - 1) \cdot (2^w - 1) + (2^w - 1) + (2^w - 1) \;=\; 2^{2w} - 1 \;<\; 2^{2 \cdot w},$$

Part 2.1: in practice (5)

Algorithms for $x \in \{0, 1, \ldots, 2^w - 1\}$

- Step #1: limb-focused operations, e.g., "multiply-accumulate with carry".

Listing

```
1 #define LIMB_MUL2(r_1,r_0,e,f,g,h) {                    \
2   dlimb_t __t  = ( dlimb_t )( e ) *                     \
3                  ( dlimb_t )( f ) +                     \
4                  ( dlimb_t )( g ) +                     \
5                  ( dlimb_t )( h ) ;                     \
6                                                         \
7   r_0 =       ( limb_t )( __t >> 0                 ); \
8   r_1 =       ( limb_t )( __t >> BITSOF( limb_t ) ); \
9 }
```

Part 2.1: in practice (5)

Algorithms for $x \in \{0, 1, \ldots, 2^w - 1\}$

- Step #1: limb-focused operations, e.g., "multiply-accumulate with carry".

Listing

```
1  #define LIMB_MUL2(r_1,r_0,e,f,g,h) {      \
2    asm( "movl %2,%%eax ; mull %3        ;  \
3          addl %4,%%eax ; adcl $0,%%edx ;  \
4          addl %5,%%eax ; adcl $0,%%edx ;  \
5          movl %%eax,%0 ; movl %%edx,%1 ;" \
6                                           \
7         : "=&g" (r_0), "=&g" (r_1)        \
8         :   "1" (e),     "r" (f),         \
9             "r" (g),     "0" (h)          \
10        : "%eax", "%edx", "cc"        );  \
11 }
```

- Step #2: $\mathbb{N}$-focused operations, e.g., "long multiplication".
  - the idea is to support computation of

x	=				6	2	3	
y	=				5	6	7	$\times$
$p_0 = 7 \cdot 3$	=					2	1	
$p_1 = 7 \cdot 2$	=				1	4		
$p_2 = 7 \cdot 6$	=			4	2			
$p_3 = 6 \cdot 3$	=				1	8		
$p_4 = 6 \cdot 2$	=			1	2			
$p_5 = 6 \cdot 6$	=		3	6				
$p_6 = 5 \cdot 3$	=			1	5			
$p_7 = 5 \cdot 2$	=		1	0				
$p_8 = 5 \cdot 6$	=	3	0					
c	=	0	1	1	1	0	0	
r	=	3	5	3	2	4	1	

  - the result r has at most $l_x + l_y$ limbs.

- Step #2: $\mathbb{N}$-focused operations, e.g., "long multiplication".

Algorithm ($\mathbb{N}$-Mul)

Input: Two unsigned, base-b integers x and y
Output: An unsigned, base-b integer $r = x \cdot y$

1 $l_x \leftarrow |x|, l_y \leftarrow |y|, l_r \leftarrow l_x + l_y$
2 $r \leftarrow 0$
3 **for** $j = 0$ **upto** $l_y - 1$ **step** $+1$ **do**
4 $\quad c \leftarrow 0$
5 $\quad$ **for** $i = 0$ **upto** $l_x - 1$ **step** $+1$ **do**
6 $\quad\quad u \cdot b + v = t \leftarrow y_j \cdot x_i + r_{j+i} + c$
7 $\quad\quad r_{j+i} \leftarrow v$
8 $\quad\quad c \leftarrow u$
9 $\quad$ **end**
10 $\quad r_{j+l_x} \leftarrow c$
11 **end**
12 **return** r

Listing

```
1  void mpn_mul( limb_t* r, const limb_t* x, int l_x,
2                           const limb_t* y, int l_y ) {
3
4    int l_r = l_x + l_y;
5
6    limb_t R[ l_r ], c;
7
8    memset( R, 0, l_r * SIZEOF( limb_t ) );
9
10   for( int j = 0; j < l_y; j++ ) {
11     c = 0;
12
13     for( int i = 0; i < l_x; i++ ) {
14       limb_t d_y = y[ j     ];
15       limb_t d_x = x[     i ];
16       limb_t d_R = R[ j + i ];
17
18       LIMB_MUL2( c, R[ j + i ], d_y, d_x, d_R, c );
19     }
20
21     R[ j + l_x ] = c;
22   }
23
24   memcpy( r, R, l_r * SIZEOF( limb_t ) );
25 }
```

- Step #3: $\mathbb{Z}$-focused operations.

Algorithm ($\mathbb{Z}$-Mul)

Input: Two signed, base-b integers x and y
Output: A signed, base-b integer $r = x \cdot y$

1 **if** $x < 0$ **and** $y \geq 0$ **then**
2 $\quad r \leftarrow -\mathbb{N}\text{-Mul}(\text{abs}(x), \text{abs}(y))$
3 **end**
4 **else if** $x \geq 0$ **and** $y < 0$ **then**
5 $\quad r \leftarrow -\mathbb{N}\text{-Mul}(\text{abs}(x), \text{abs}(y))$
6 **end**
7 **else if** $x \geq 0$ **and** $y \geq 0$ **then**
8 $\quad r \leftarrow +\mathbb{N}\text{-Mul}(\text{abs}(x), \text{abs}(y))$
9 **end**
10 **else if** $x < 0$ **and** $y < 0$ **then**
11 $\quad r \leftarrow +\mathbb{N}\text{-Mul}(\text{abs}(x), \text{abs}(y))$
12 **end**
13 **return** r

Listing

```
1 void mpz_mul( mpz_t* r, const mpz_t* x,
2                         const mpz_t* y ) {
3   if( x->s == y->s ) {
4     MPZ_MUL( r, x, y, MPZ_SIGN_POS );
5   }
6   else {
7     MPZ_MUL( r, x, y, MPZ_SIGN_NEG );
8   }
9 }
```

- Step #3: $\mathbb{Z}$-focused operations.

Listing

```
1 #define MPZ_MUL(r,x,y,s_r) {                                    \
2   int l_r = (x)->l + (y)->l;                                    \
3                                                                 \
4                       mpn_mul( (r)->d, (x)->d, (x)->l,          \
5                                        (y)->d, (y)->l );        \
6                                                                 \
7   (r)->l              = mpn_lop( (r)->d, l_r );                 \
8   (r)->s              = s_r;                                    \
9 }
```

Listing

```
1 int mpn_lop( const limb_t* x, int l_x ) {
2   while( ( l_x > 1 ) && ( x[ l_x - 1 ] == 0 ) ) {
3     l_x--;
4   }
5
6   return l_x;
7 }
```

- Problem:
  - given the **base** $x \in \mathbb{Z}_N^\times$ and the **exponent** $y \in \mathbb{Z}$, we want to compute $r = x^y \pmod N$,
  - we could do so via

$$r = x^y = \underbrace{x \times x \times \cdots \times x}_{y \text{ terms}} \pmod N,$$

    but this is $O(y)$ and the magnitude of y can be very large.
- Solution:
  - algorithms for efficient

multiplicative group $\mathbb{G}^\times$	$\rightsquigarrow$	**exponentiation**	x^y
additive group $\mathbb{G}^+$	$\rightsquigarrow$	**scalar multiplication**	$[y]\,x$

Part 2.2: in practice (2)

- Idea: consider some $\mathbb{G}^{\times}$.
  - exponentiation *means* repeated application of the group operation, i.e.,

$$x^y = \underbrace{x \times x \times \cdots \times x}_{y \text{ terms}},$$

  so if $y = 14_{(10)}$ we have

$$x^y = x \times x \times x \times x \times x \times x \times x \times x \times x \times x \times x \times x \times x \times x.$$

  - expressing y in base-2, we can rewrite this as

$$\begin{aligned} x^y &= x^{\sum_{i=0}^{n-1} y_i \cdot 2^i} \\ &= x^{y_{n-1} \cdot 2^{n-1} + \cdots + y_1 \cdot 2^1 + y_0 \cdot 2^0} \\ &= x^{y_{n-1} \cdot 2^{n-1}} \times \cdots \times x^{y_1 \cdot 2^1} \times x^{y_0 \cdot 2^0} \end{aligned}$$

- Idea: consider some $\mathbb{G}^{\times}$.
  - given $y = 14_{(10)} = 1110_{(2)}$ we can see that

$$\begin{aligned} x^y &= x^{y_0 \cdot 2^0} \times x^{y_1 \cdot 2^1} \times x^{y_2 \cdot 2^2} \times x^{y_3 \cdot 2^3} \\ &= x^{0 \cdot 2^0} \times x^{1 \cdot 2^1} \times x^{1 \cdot 2^2} \times x^{1 \cdot 2^3} \\ &= x^0 \times x^2 \times x^4 \times x^8 \\ &= x^{14} \end{aligned}$$

  - given $y = 14_{(10)} = 1110_{(2)}$ we can see that

$$\begin{aligned} x^y = {} & x^{y_0} \times \left(x^{y_1} \times \left(x^{y_2} \times \left(x^{y_3} \times \left(1 \right)^2 \right)^2 \right)^2 \right)^2 \\ & x^0 \times \left(x^1 \times \left(x^1 \times \left(x^1 \times \left(1 \right)^2 \right)^2 \right)^2 \right)^2 \\ & x^0 \times \left(x^1 \times \left(x^1 \times \left(x^1 \times 1 \right)^2 \right)^2 \right)^2 \\ & x^0 \times \left(x^1 \times \left(x^1 \times \left(x^1 \right)^2 \right)^2 \right)^2 \\ & x^0 \times \left(x^1 \times \left(x^1 \times x^2 \right)^2 \right)^2 \\ & x^0 \times \left(x^1 \times \left(x^3 \right)^2 \right)^2 \\ & x^0 \times \left(x^1 \times x^6 \right)^2 \\ & x^0 \times \left(x^7 \right)^2 \\ & x^0 \times x^{14} \\ & x^{14} \end{aligned}$$

    via application of **Horner's rule**.

- Option #1: left-to-right *binary* exponentiation.
  - evaluate the Horner expansion step-by-step in an "inside-out" order ,
  - express y in base-2, i.e., extract a 1-bit digit d from y in each step,
  - trade-off time in favour of space,
  - apply

$$r \leftarrow \begin{cases} r^2 & \text{if } d = 0 \\ r^2 \times x & \text{if } d = 1 \end{cases}$$

  so as to *accumulate* the result.

Part 2.2: in practice (3)

- Option #1: left-to-right *binary* exponentiation.

Algorithm (1Exp-L2R-Binary)

Input: A group element $x \in \mathbb{G}^{\times}$, a base-2 integer $0 \leq y < n$
Output: The group element $r = x^y \in \mathbb{G}^{\times}$

```
1 r ← 1
2 for i = |y| − 1 downto 0 step −1 do
3 |   r ← r^2
4 |   if y_i = 1 then
5 |   |   r ← r × x
6 |   end
7 end
8 return r
```

- Option #2: left-to-right *windowed* exponentiation.
  - evaluate the Horner expansion step-by-step in an "inside-out" order ,
  - express y in base-m for $m = 2^k$, i.e., extract a k-bit digit d from y in each step,
  - trade-off space in favour of time, i.e., perform pre-computation reflected by X,
  - apply

$$r \leftarrow \begin{cases} r^{2^k} & & & & \text{if } d = 0 \\ r^{2^k} & \times & (\, X[0] & = x^1 \,) & \text{if } d = 1 \\ r^{2^k} & \times & (\, X[1] & = x^2 \,) & \text{if } d = 2 \\ & \vdots & & & \\ r^{2^k} & \times & (\, X[2^k - 2] & = x^{2^k - 1} \,) & \text{if } d = 2^k - 1 \end{cases}$$

  so as to *accumulate* the result.

- Option #2: left-to-right *windowed* exponentiation.

Algorithm (Recode-Base-m)

Input: An integer y represented in base-2, an integer $m = 2^k$
Output: A sequence y' which represents y in base-2^k

$y' \leftarrow \emptyset, i \leftarrow 0$
while $y \neq 0$ **do**
  $y'_i \leftarrow y \wedge (2^k - 1), y \leftarrow y \gg k, i \leftarrow i + 1$
end
return y'

Algorithm (1Exp-L2R-FixedWindow)

Input: A group element $x \in \mathbb{G}^\times$, a base-2 integer $0 \leq y < n$, an integer $m = 2^k$
Output: The group element $r = x^y \in \mathbb{G}^\times$

$y' \leftarrow$ Recode-Base-$m(y, m = 2^k)$
Pre-compute $X = [x^i \mid i \in \{1, 2, 3, \ldots, m - 1 = 2^k - 1\}]$
$r \leftarrow 1$
for $i = |y'| - 1$ **downto** 0 **step** -1 **do**
  for $j = 0$ **upto** $k - 1$ **step** $+1$ **do**
    $r \leftarrow r^2$
  end
  if $y'_i \neq 0$ **then**
    $r \leftarrow r \times X[y'_i - 1]$
  end
end
return r

Part 2.3: in practice (1)

- Problem:
  - we want to represent and perform operations on integers modulo N, i.e., elements of
$$\mathbb{Z}_N = \{0, 1, 2, \ldots, N-1\},$$
  - we *could* implement $\mathbb{Z}_N$ based on $\mathbb{Z}$, but any $x \in \mathbb{Z}_N$ will
    - always be positive, and
    - have a size that is upper-bounded by N.
- Solution:
  1. a data structure to represent x, and
  2. algorithms to operate on instances of said data structure.

Part 2.3: in practice (1)

- Problem:
  - we want to represent and perform operations on integers modulo N, i.e., elements of
$$\mathbb{Z}_N = \{0, 1, 2, \ldots, N-1\},$$
  - we *could* implement $\mathbb{Z}_N$ based on $\mathbb{Z}$, but any $x \in \mathbb{Z}_N$ will
    - always be positive, and
    - have a size that is upper-bounded by N.
- Solution:
  1. use the existing data structure for $\mathbb{N}$, and
  2. focus on an algorithm for $x \times y \pmod{N}$ to support $\mathbb{Z}_N^{\times}$.

- Idea:
  1. Use the relationship

$$t \pmod N \equiv t - (N \times \left\lfloor \frac{t}{N} \right\rfloor)$$

     which
     - uses a standard integer representation,
     - requires no pre-computation, but
     - requires an integer division, which is relatively complicated and computationally expensive.
  2. Use **Barrett reduction** [4], which
     - uses a standard integer representation,
     - requires some pre-computation, and
     - requires $2 \cdot (l_N + 1) \cdot (l_N + 1)$ limb multiplications for an l_N-limb N.
  3. Use **Montgomery reduction** [7], which
     - uses a non-standard integer representation,
     - requires some pre-computation, and
     - requires $2 \cdot l_N \cdot l_N$ limb multiplications for an l_N-limb N.

Part 2.3: in practice (3)

- Montgomery-based multiplication $\rightsquigarrow$ exponentiation $\rightsquigarrow$ RSA:
  - pre-compute a set of **Montgomery parameters** $\Pi = (N, \rho, \omega)$ where
    1. $\rho = b^k$ for the smallest k such that $b^k > N$, and
    2. $\omega = -N^{-1} \pmod{\rho}$,

    assuming a base-b representation of N such that $\gcd(N, b) = 1$,

- Montgomery-based multiplication $\rightsquigarrow$ exponentiation $\rightsquigarrow$ RSA:

  - the **Montgomery representation** of an integer

    $$0 \leq x < N$$

    is then defined as

    $$\hat{x} = x \times \rho \pmod{N},$$

- Montgomery-based multiplication $\rightsquigarrow$ exponentiation $\rightsquigarrow$ RSA:

- using these concepts, we can define **Montgomery multiplication**:

Algorithm (MontMul)

Input: A set of Montgomery parameters $\Pi = (N, \rho, \omega)$, intgers $0 \leq x, y < N$
Output: $r = x \times y \times \rho^{-1} \pmod{N}$

```
1 r ← x × y
2 r ← (r + ((r × ω) mod ρ) × N)/ρ
3 if r >= N then r ← r − N
4 return r
```

where, crucially and by-design, the modular reduction and division by ρ are special-case.

- Montgomery-based multiplication $\rightsquigarrow$ exponentiation $\rightsquigarrow$ RSA:
  - given we can convert into

$$\begin{aligned} \textsc{MontMul}((N, \rho, \omega), x, \rho^2 \bmod N) &= x \times \rho^2 \times \rho^{-1} \pmod N \\ &= x \times \rho \pmod N \\ &= \hat{x} \end{aligned}$$

    and from

$$\begin{aligned} \textsc{MontMul}((N, \rho, \omega), \hat{x}, 1) &= \hat{x} \times 1 \times \rho^{-1} \pmod N \\ &= (x \times \rho) \times 1 \times \rho^{-1} \pmod N \\ &= x \end{aligned}$$

    Montgomery representation,
  - we *could* then compute

$$r = x \times y \pmod N \quad \leftrightarrow \quad \begin{cases} \hat{x} &= \textsc{MontMul}((N, \rho, \omega), x, \rho^2 \bmod N) \\ \hat{y} &= \textsc{MontMul}((N, \rho, \omega), y, \rho^2 \bmod N) \\ \hat{r} &= \textsc{MontMul}((N, \rho, \omega), \hat{x}, \hat{y}) \\ r &= \textsc{MontMul}((N, \rho, \omega), \hat{r}, 1) \end{cases}$$

    but the overhead of conversion is too high ...

Part 2.3: in practice (3)

- Montgomery-based multiplication $\rightsquigarrow$ exponentiation $\rightsquigarrow$ RSA:
  - ... instead, we adapt an exponentiation algorithm: for example

Algorithm (MontExp)

Input: A set of Montgomery parameters $\Pi = (N, \rho, \omega)$, intgers $0 \leq x, y < N$
Output: $r = x^y \pmod N$

```
1  r̂ ← MontMul(Π, 1, ρ² mod N), x̂ ← MontMul(Π, x, ρ² mod N)
2  for i = |y| − 1 downto 0 step −1 do
3  |   r̂ ← MontMul(Π, r̂, r̂)
4  |   if y_i = 1 then
5  |   |   r̂ ← MontMul(Π, r̂, x̂)
6  |   end
7  end
8  return MontMul(Π, r̂, 1)
```

i.e.,

- convert input into Montgomery representation,
- compute a series of (many) Montgomery multiplications,
- convert output from Montgomery representation,

and thus amortise the conversion.

Conclusions

- Take away points: you can often simple *use*

$$c = \mathsf{RSA.Enc}((N, e), m) = m^e \pmod{N},$$

but understanding internals of this primitive can be useful and/or important.

- some historically interesting aspects; some "portable" concepts,
- close relationship between primitive and underlying Mathematics,
- wide range of viable implementation strategies,
- extensive deployment, in various contexts and use-cases.

Additional Reading

- *Wikipedia: RSA*. URL: https://en.wikipedia.org/wiki/RSA.
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